Rational Sentiments and Financial Frictions*

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Abstract

We provide a complete analysis of previously undocumented sunspot equilibria in a canonical dynamic economy with imperfect risk sharing. Methodologically, we employ stochastic stability theory to establish existence of this broad class of sunspot equilibria. Economically, self-fulfilling fluctuations are characterized by uncertainty shocks: changing beliefs about volatility trigger asset trades, which impacts productive efficiency and justifies the degree of uncertainty. We show how rational sentiment helps resolve two puzzles in the macro-finance literature: (i) financial crises emerge suddenly, featuring (quantitatively) hard-to-explain volatility spikes and asset-price declines; (ii) asset-price booms, with below-average risk premia, predict busts and financial crises.

JEL Codes: E00, E44, G01.

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It has by now become commonplace, especially after the 2008 global financial crisis, for macroeconomic models to prominently feature banks, limited participation, imperfect risk-sharing, and other such "financial frictions." Incorporating these features allows macroeconomists to speak meaningfully about financial crises and desirable policy responses. Despite the dramatic growth in this literature, there remain two major sources of disconnect between these models and actual data. For one, standard models have difficulty reproducing the observed severity and suddenness of economic downturns and asset-price dislocations. Secondly, standard models struggle to generate booms that are inherently fragile and prone to bust. To address these shortcomings, some recent contributions add large and sudden bank runs[1] while others deviate from rational expectations to model booms as episodes of over-optimism.[2]

We embrace *rational sentiment* as a complementary approach. This paper makes two main contributions. First, we uncover a wide variety of novel sentiment-driven sunspot equilibria supported by standard financial friction models. The fluctuations in these equilibria are self-fulfilling: they only occur because agents expect them and coordinate on them. Second, we demonstrate how sentiment fluctuations alleviate some of the empirical shortcomings for this class of models. Rational sentiment can generate both (i) large and sudden fluctuations, similar to bank runs (footnote 1), and (ii) booms that breed fragility, similar to the "behavioral sentiment" adopted by some recent papers (footnote 2).

Model and mechanism. We study a simple stripped-down model with financial frictions, similar to Kiyotaki and Moore (1997), Brunnermeier and Sannikov (2014), and many others.[3] There are two types of agents ("experts" and "households") with identical preferences but different levels of productivity when managing capital. Heterogeneous productivity means the identity of capital holders matters for aggregate output. But incomplete markets prevent agents from sharing risks associated to their capital hold-

[1] For example, Gertler and Kiyotaki (2015) and Gertler et al. (2020) attempt to integrate bank runs into a conventional financial accelerator model, in order to capture additional amplification and non-linearity. These runs are assumed to be large aggregate phenomena in the sense that the entire banking system suddenly collapses. Without runs or panic-like behavior, financial accelerator models have a difficult time inducing the financial intermediary leverage and risk concentration needed to generate large amounts of amplification. This shortcoming can be seen in Di Tella (2017), where the retirement rate of bankers is calibrated to 115% per year, or in Khorrami (2018), where the implied entry costs needed to match asset price dynamics are on the order of 90% of wealth.

[2] For example, Krishnamurthy and Li (2020) and Maxted (2020) build an extrapolative sentiment process on top of a relatively standard financial accelerator model. Agents' excessive optimism in booms lowers risk premia, erodes bank balance sheets, and creates fragility.

[3] We work in continuous time, contributing to a burgeoning literature (He and Krishnamurthy, 2012, 2013, 2019; Moreira and Savov, 2017; Klimenko et al., 2017; Caballero and Simsek, 2020).

ings, so optimal capital holdings depend to some degree on these risks and not only on productivities. There are no other features: no borrowing/collateral constraints, no default externalities, and no irrational beliefs. And yet, this basic model can feature a tremendous amount of multiplicity that has been overlooked in the literature.

The following story clarifies the mechanism. Suppose there is a sudden rise in *fear*, purely as a sunspot phenomenon (i.e., no change in preferences or technology). Fear manifests as higher perceived asset-price volatility, which results in a *fire sale*: first-best capital users (experts) sell to less-efficient users (households). The reason is that, with financial frictions, both risk-sharing and productive-efficiency considerations matter for the capital distribution, and elevated risk propels risk-sharing considerations to the forefront. Because productive efficiency falls, asset prices fall as well. Thus, the new allocation features a less efficient capital allocation, lower asset prices, and higher volatility.

This allocation will only be an equilibrium if it does not lead to explosive paths. While this may seem technical, it is a real concern: with higher volatility in the new equilibrium, any subsequent fear shocks would have a larger direct impact, further raise volatility, and so on, ad infinitum. Rational forward-looking agents would rule this out at the beginning and suppress their fear.

In this class of models, explosions are easily prevented because of an indeterminacy in the dynamic equilibrium. The key observation is that optimal capital holdings are a function of the *risk premium*. Consequently, only the risk premium is pinned down by equilibrium; risky expected returns and riskless rates are not separately determined. This indeterminacy in expected returns, hence expected capital gains, provides a tool with which sunspot equilibria can be engineered.

As long as agents expect asset prices to "bounce back" from "extreme values," short-run fear can be consistent with non-explosive long-run equilibrium. Nothing rules out such bounce-back beliefs in this class of models. In our continuous-time setup, bounce-back beliefs are boundary conditions on expected returns at the extreme states. Such boundary restrictions are both analytically-convenient and mild; nearly arbitrary dynamics are possible away from extreme states.

In summary, a sunspot rise in fear creates a self-fulfilled decline in asset prices, through coordinated fire sales. Conversely, sunspot bravery (decline in fear) raises asset prices, through coordinated purchases.

Overview of paper. While explaining our mechanism above, we abstracted from the wealth distribution between experts and households. Typically in the financial frictions literature, this wealth distribution is the key state variable modulating the dynamics.

The first results of our paper (Section 2) demonstrate how restricting attention to these type of equilibria—equilibria which are Markovian solely in the wealth distribution—precludes essentially all interesting self-fulfilling dynamics.

Our main results pertain to a richer class of self-fulfilling equilibria (Section 3). Mathematically, we dispense with the assumption that equilibria must be Markovian in the wealth distribution, which can be understood as removing an ad-hoc restriction on agents' beliefs. This generalization considerably complicates the analysis, and our contribution here is to provide an explicit construction and characterization of a broad class of such equilibria.

This richer class of equilibria engender several new insights, related to the shortcomings discussed earlier (Section 4). First, whereas fundamentals-based recessions, which are primarily about bank balance sheet impairment in our model, feature small volatility increases and very slow recoveries, sentiment-driven crises feature far larger volatility spikes and fast recoveries. In fact, we prove that arbitrarily large capital price volatility and arbitrary recovery speeds can be justified by sunspot equilibria. Second, whereas fundamentals-based booms always reduce the prospect of crisis, sentiment-driven booms can actually increase crisis probabilities. Relatedly, in the years before large busts, an economy with sentiment tends to feature asset-price and output booms, low volatility, and below-average risk premia. We argue all of these properties of sentiment-driven fluctuations better resemble real-world financial cycles.

Related literature. The theoretical focus on financial frictions and sunspots is not new to this paper. Several studies show how multiplicity emerges through the interaction between asset valuations and borrowing constraints.[4] Relative to these papers, we study different and more primitive financial frictions (equity-issuance constraints) that do not feature any mechanical link between prices and constraints.[5]

Bank runs, financial panics, and sudden stops are related to, but distinct from, our self-fulfilled fluctuations.[6] All of these phenomena rely on financial frictions, are outcomes of coordination, and produce large fluctuations relative to fundamentals. How-

[4]For instance, bubbles can relax credit constraints, allowing greater investment and thus justifying the existence of the bubble (Scheinkman and Weiss, 1986; Kocherlakota, 1992; Farhi and Tirole, 2012; Miao and Wang, 2018; Liu and Wang, 2014). Self-fulfilling credit dynamics can also arise with *unsecured* lending as opposed to collateralized (Gu et al., 2013; Azariadis et al., 2016).

[5]We call equity-issuance constraints "more primitive" because they are present (either explicitly or implicitly) even in models with borrowing constraints. With unlimited outside equity, perfect risk-sharing could always be achieved and the effects of borrowing constraints circumvented.

[6]In a setup close to ours, Mendo (2020) studies self-fulfilled panics that induce collapse of the financial sector, an extreme example of the fluctuations we analyze. Gertler and Kiyotaki (2015) and Gertler et al. (2020) study bank runs in a similar class of models.

ever, whereas bank runs and its cousins are liability-side phenomena, self-fulfilled fire sales are pure asset-side phenomena.[7] Furthermore, unlike runs, our mechanism does not require asset-market illiquidity or maturity mismatch. Finally, whereas runs are almost exclusively about large downside risk, our sentiment fluctuations also generate interesting boom-bust cycles.

Given our results hold even without borrowing constraints or runs, we illustrate that a much broader class of financial crisis models are subject to sunspots. We also do not rely on the more traditional multiplicity-inducing assumptions, such as overlapping generations, non-convexities or externalities in technology,[8] asymmetry of information,[9] or multiple assets.[10]

Finally, our equilibrium construction differs deeply from the literature. Sunspot equilibria are often constructed by essentially randomizing over a multiplicity of deterministic transition paths to a stable steady state. By contrast, the deterministic version of our model features an unstable steady state; critically, the introduction of volatility flips the stability properties of equilibrium. This distinction is likely why our sunspot equilibria have gone unnoticed despite the framework being so widespread. Methodologically, we prove our existence results with tools from the "stochastic stability" literature (the stochastic differential equation analog of Lyapunov stability for ODE systems). As one might expect from deterministic models, the existence of sunspot equilibria is tied directly to stability properties. Stochastic stability tools are ideally suited for this issue.

1 Model

Information Structure. Time $t \geq 0$ is continuous. There are two types of uncertainty in the economy, modeled as two independent Brownian motions $Z := (Z^{(1)}, Z^{(2)})$. All random processes will be adapted to Z.[11] As will be clear below, the first shock $Z^{(1)}$

[7] When selling assets, investors simultaneously deleverage, which clarifies our mechanism as a "funding demand" decline rather than the "funding supply" decline that characterizes a run. It is not that investors cannot obtain financing, just that they do not want to.

[8] For example, see Azariadis and Drazen (1990) for multiplicity under threshold investment behavior. See Farmer and Benhabib (1994) for a multiplicity under increasing returns to scale.

[9] In a macro context, Piketty (1997) and Azariadis and Smith (1998) for self-fulfilling dynamics in the presence of screened/rationed credit. In a finance context, Benhabib and Wang (2015) and Benhabib et al. (2016, 2019) generate sunspot fluctuations in dispersed information models.

[10] Hugonnier (2012), Gârleanu and Panageas (2021), and Khorrami and Zentefis (2020) all build "redistributive" sunspots that shift valuations among multiple positive-net-supply assets.

[11] In the background, the Brownian motion Z exists on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, equipped with all the "usual conditions." All equalities and inequalities involving random variables are understood to hold almost-everywhere and/or almost-surely.

represents a *fundamental shock* in the sense that it directly impacts production possibilities, whereas the second shock $Z^{(2)}$ is a *sunspot shock* that is extrinsic to any economic primitives but nevertheless may impact endogenous objects. At the end of the paper, we will also consider extrinsic Poisson jumps as part of the information structure.

Technology, Markets. There are two goods, a non-durable good (the numéraire, "consumption") and a durable good ("capital") that produces the consumption good. The aggregate supply of capital grows exogenously as

$$dK_t = K_t[gdt + \sigma dZ_t^{(1)}],$$

where g and σ are exogenous constants. The capital-quality shock $\sigma dZ^{(1)}$ is a standard way to introduce fundamental randomness in technology. Individual capital holdings evolve identically.

There are two types of agents, experts and households, who differ in their production technologies. Experts produce a_e units of the consumption good per unit of capital, whereas households' productivity is $a_h \in (0, a_e)$.

Capital is freely tradable, with its relative price denoted by q_t, determined in equilibrium. Conjecture the following form for capital price dynamics:

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t} \cdot dZ_t]. \tag{1}$$

There are two potential avenues for random fluctuations. The standard term $\sigma_q \cdot \binom{1}{0}$ represents amplification (or dampening) of fundamental shocks, as in Brunnermeier and Sannikov (2014) and others. By contrast, $\sigma_q \cdot \binom{0}{1}$ measures sunspot volatility that only exists because agents believe in it.

Financial markets consist solely of an instantaneously-maturing, risk-free bond that pays interest rate r_t is in zero net supply. The key financial friction: agents cannot issue equity when managing capital. It is inconsequential that the constraint be this extreme. Partial equity issuance, as long as there is some limit, will generate identical results on sunspot volatility.[12]

Preferences and Optimization. Given the stated assumptions, we can write the dynamic

[12] In particular, a partial equity-issuance constraint simply scales the mapping between expert wealth and asset prices. As is well-known, the equilibrium of economies in the class we consider will live in the region where the equity constraint is always-binding. Equity-issuance restrictions, sometimes called "skin-in-the-game" constraints, often arise as the optimal contract in a moral hazard problem, though this micro-foundation is not important for our purposes here.

budget constraint of an agent of type j (expert or household) as

$$dn_{j,t} = \Big[(n_{j,t} - q_t k_{j,t}) r_t - c_{j,t} + a_j k_{j,t}\Big] dt + d(q_t k_{j,t}), \tag{2}$$

where n_j is the agent's net worth, c_j is consumption, and k_j is capital holdings. The term $d(qk)$ represents the capital and price appreciation that accrues while holding capital.

Experts and households have time-separable logarithmic utility, with discount rates ρ_e and $\rho_h \leq \rho_e$, respectively. All agents have rational expectations and solve

$$\sup_{c_j \geq 0,\, k_j \geq 0,\, n_j \geq 0} \mathbb{E}\Big[\int_0^\infty e^{-\rho_j t} \log(c_{j,t}) dt\Big] \tag{3}$$

subject to (2). The constraint $n_{j,t} \geq 0$ is the standard solvency constraint. Everything in this optimization problem is homogeneous in (c, k, n), so we can think of the expert and household as representative agents within their class.

Finally, to guarantee a stationary wealth distribution, we also allow an overlapping generation structure: agents perish idiosyncratically at rate δ; perishing agents are replaced by newborns, who inherit an equal share of perishing wealth; a fraction $\nu \in [0, 1]$ of newborns are exogenously designated experts, and $1 - \nu$ are households; there are no annuity markets to trade death risk. As the death rate δ affects an agent's lifetime utility, the subjective discount rates ρ_e, ρ_h are assumed inclusive of δ. To acknowledge the fact that OLG creates intertemporal transfers across agent types, which do not affect alive agents' individual net worth evolution, let N_e and N_h denote aggregate expert and household net worth. The dynamic evolutions of N_e and N_h will mirror (2), with additional terms capturing OLG-related transfers.[13] We reiterate that OLG is unnecessary for our sunspot results and only serves to obtain stationarity in case we set $\rho_e = \rho_h$.

Equilibrium. The definition of competitive equilibrium is standard: (i) taking prices as given, and given exogenous time-0 allocations of capital and riskless bonds, experts and households solve (3) subject to (2); (ii) consumption and capital markets clear at all dates:

$$c_{e,t} + c_{h,t} = a_e k_{e,t} + a_h k_{h,t} \tag{4}$$

$$k_{e,t} + k_{h,t} = K_t. \tag{5}$$

The riskless bond market clears automatically by Walras' Law.

[13] I.e., $dN_e = N_e \frac{dn_e}{n_e} - \delta N_e dt + \delta\nu(N_e + N_h)dt$ and $dN_h = N_h \frac{dn_h}{n_h} - \delta N_h dt + \delta(1-\nu)(N_e + N_h)dt$.

To benchmark this environment, note that frictionless equity issuance allows perfect risk-sharing and efficient production ($k_h = 0$). In this frictionless world, there can be no sunspot volatility nor amplification ($\sigma_q = 0$). As is well known, limited equity issuance begets imperfect risk-sharing and inefficient production, opening the door for amplification; our contribution is to show sunspot volatility can also emerge.

Characterization of Equilibrium. We present a useful equilibrium characterization that aids all future analysis. Given log utility and the scale-invariance of agents' budget sets, individual optimization problems are readily solvable. Optimal consumption satisfies the standard formula $c_j = \rho_j n_j$. Optimal capital holding by experts and households implies

$$\frac{a_e}{q} + g + \mu_q + \sigma\sigma_q \cdot \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) - r = \frac{qk_e}{n_e}|\sigma_R|^2 \tag{6}$$

$$\frac{a_h}{q} + g + \mu_q + \sigma\sigma_q \cdot \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) - r \leq \frac{qk_h}{n_h}|\sigma_R|^2 \quad \text{(with equality if } k_h > 0\text{)}, \tag{7}$$

where

$$\sigma_{R,t} := \sigma\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + \sigma_{q,t} \tag{8}$$

denotes the shock exposure of capital returns.

Next, we aggregate. Due to financial frictions and productivity heterogeneity, both the distribution of wealth and capital holdings will matter in equilibrium. However, because all experts (and households) make the same scaled consumption c_j/n_j and portfolio choices k_j/n_j, the wealth and capital distributions may be summarized by experts' wealth share $\eta := N_e/(N_e + N_h) = N_e/qK$ and experts' capital share $\kappa := k_e/K$. Substitute optimal consumption into goods market clearing (4), and use the definitions of η and κ, to obtain

$$q\bar{\rho} = \kappa a_e + (1 - \kappa)a_h, \tag{PO}$$

where $\bar{\rho}(\eta) := \eta\rho_e + (1 - \eta)\rho_h$ is the wealth-weighted average discount rate. Equation (PO) connects asset price q to output efficiency κ, which we call a *price-output* relation for short.

Differencing the optimal portfolio conditions (6)-(7), we obtain the *risk balance* condition

$$0 = \min\left[1-\kappa,\ \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}|\sigma_R|^2\right]. \tag{RB}$$

Either experts manage the entire capital stock ($\kappa = 1$) or the excess return experts obtain over households, $(a_e - a_h)/q$, represents fair compensation for differential risk exposure, $\frac{\kappa-\eta}{\eta(1-\eta)}|\sigma_R|^2$. On the other hand, summing agents' portfolio optimality conditions (weighted by κ and $1-\kappa$) yields an equation for the riskless rate:

$$r = \frac{\kappa a_e + (1-\kappa)a_h}{q} + g + \mu_q + \sigma\sigma_q \cdot \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) - \left(\frac{\kappa^2}{\eta} + \frac{(1-\kappa)^2}{1-\eta}\right)|\sigma_R|^2. \tag{9}$$

Finally, by time-differentiating the definition of experts' wealth share $\eta = N_e/(N_e + N_h)$, and using agents' net worth dynamics (2) along with contributions from OLG, wealth share dynamics are given by

$$d\eta_t = \mu_{\eta,t}dt + \sigma_{\eta,t} \cdot dZ_t, \quad \text{given} \quad \eta_0, \tag{10}$$

where

$$\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + (\kappa - 2\eta\kappa + \eta^2)\frac{\kappa - \eta}{\eta(1-\eta)}|\sigma_R|^2 + \delta(\nu - \eta) \tag{11}$$

$$\sigma_\eta = (\kappa - \eta)\sigma_R. \tag{12}$$

The preceding equations are the only ones imposed by equilibrium. We thus simplify our search for equilibria by looking for processes that satisfy them.

Definition 1. Given $\eta_0 \in (0,1)$, an *equilibrium* consists of processes $(\eta_t, q_t, \kappa_t, r_t)_{t\geq 0}$ such that equations (PO), (RB), (9), and (11)-(12) hold for all $t \geq 0$.

Remark 1 (Transversality and No-Ponzi)**.** *By additionally imposing a "No-Ponzi condition" on individual agents, we require the conditions*[14]

$$\lim_{T\to\infty} e^{-\rho_e T}\frac{\kappa_T}{\eta_T} = 0 \quad \textit{and} \quad \lim_{T\to\infty} e^{-\rho_h T}\frac{1-\kappa_T}{1-\eta_T} = 0. \tag{13}$$

[14] Here is the proof that (13) is required. Let $(M_{j,t})_{t\geq 0}$ denote the state-price density process for type-j agents, $j \in \{e, h\}$. Optimality implies $M_{j,t} = e^{-\rho_j t}/c_{j,t} = e^{-\rho_j t}/(\rho_j n_{j,t})$. Thus, $\lim_{T\to\infty} M_{j,T} n_{j,T} = 0$ (note also that the individual transversality condition $\lim_{T\to\infty}\mathbb{E}_t[M_{j,T} n_{j,T}] = 0$ is automatically satisfied). Letting $b_{j,t} := q_t k_{j,t} - n_{j,t}$ be the debt position, the No-Ponzi condition states $\lim_{T\to\infty} M_{j,T} b_{j,T} \leq 0$, and the inequality "$\leq$" is replaced by an equality "$=$" under optimality. Combining these results, we have the requirement $\lim_{T\to\infty} M_{j,T} q_T k_{j,T} = 0$, which becomes (13) after substituting $M_{j,t}$, $q_t k_{e,t}/n_{e,t} = \kappa_t/\eta_t$, and $q_t k_{h,t}/n_{h,t} = (1-\kappa_t)/(1-\eta_t)$.

In all of our equilibria, we will show that $(\eta_t)_{t\geq 0}$ possesses a stationary distribution on $(0,1)$, with no mass at the boundaries, so condition (13) will always be satisfied.

Finally, we categorize our equilibria into two types: fundamental and sunspot. Fundamental equilibria have two properties: (i) the sunspot shock $Z^{(2)}$ plays no role; and (ii) only a minimal set of state variables affects observables. Because of financial frictions and productivity heterogeneity, the expert wealth share η is a necessary state variable to summarize economic conditions. Other objects (e.g., q, r, κ) are either prices or control variables, so there is a sense in which η is the minimal state variable needed in this class of models. In other words, a fundamental equilibrium should only depend on η. Sunspot equilibria constitute all other equilibria, which we further categorize into two types depending on whether or not they are Markov in η.

Definition 2. A *Fundamental Equilibrium* (FE) is an equilibrium that is Markov in η and in which $\sigma_q \cdot \binom{0}{1} \equiv 0$. Any other equilibrium is a *Brownian Sunspot Equilibrium* (BSE). A BSE that is Markov in η is called a *Wealth-driven BSE* (W-BSE). A BSE that is non-Markov in η is called a *Sentiment-driven BSE* (S-BSE).

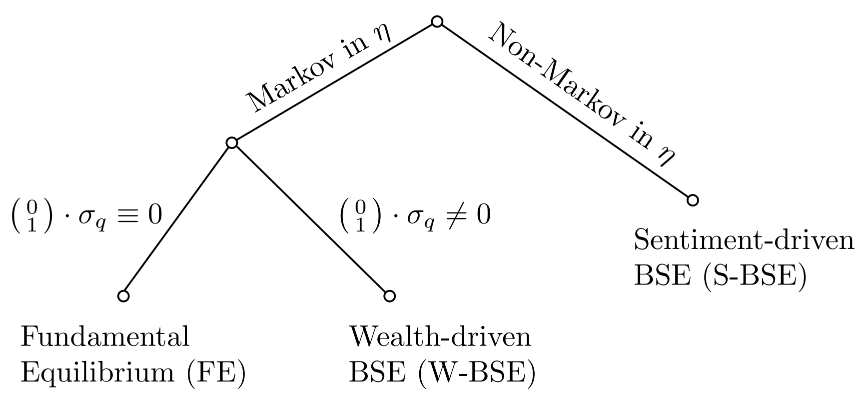

Figure 1: Types of equilibria.

Figure 1 displays the equilibrium taxonomy. We proceed as follows. Section 2 studies W-BSEs, while Sections 3-4 concern S-BSEs.

2 A class of uninteresting equilibria

Universally, papers studying this class of models restrict attention to Markov equilibria in which η is the only state variable. This section illustrates how these equilibria, even if sunspot shocks can matter, are too restrictive for our purposes. We first present a W-BSE

(Section 2.1) and then argue is is inherently uninteresting (Sections 2.2-2.3). Alternatively, readers can skip directly to Section 3 with no difficulty.

2.1 W-BSE: existence and properties

In this section, we study the version of the model without fundamental shocks, i.e., $\sigma = 0$. Hence, both shocks represent extrinsic uncertainty, and we dispense with $Z^{(1)}$ to simplify the exposition.[15] Without any intrinsic uncertainty, there always exists a deterministic Fundamental Equilibrium (FE).

Lemma 1 (Fundamental Equilibrium)**.** *If $\sigma = 0$, there exists an equilibrium in which experts manage all capital, $\kappa = 1$, and its price $q_t = a_e/\bar{\rho}(\eta_t)$ evolves deterministically.*

But there is also another equilibrium, a Wealth-driven Brownian Sunspot Equilibrium (W-BSE), which is Markov in η and has volatility. In this W-BSE, the capital price will depend only on η, i.e., $q_t = q(\eta_t)$ for some function q. By Itô's formula, we then have $\sigma_q = \frac{q'}{q}\sigma_\eta$. On the other hand, equations (8) and (12) with $\sigma = 0$ imply $\sigma_\eta = (\kappa - \eta)\sigma_q$. Solving this two-way feedback between σ_q and σ_η,

$$\left[1 - (\kappa - \eta)\frac{q'}{q}\right]\sigma_q = 0. \tag{14}$$

There are two possibilities: either (i) $\sigma_q = 0$, which corresponds to the FE of Lemma 1; or (ii) $1 = (\kappa - \eta)\frac{q'}{q}$, in which case σ_q can be non-zero. We pursue the latter.

Substituting $\kappa < 1$ from (PO), we obtain a first-order ODE for q:

$$q' = \frac{(a_e - a_h)q}{q\bar{\rho} - \eta a_e - (1-\eta)a_h}, \quad \text{if} \quad \kappa < 1. \tag{15}$$

Consider boundary condition $\kappa(0) = 0$, which translates via (PO) to $q(0) = a_h/\rho_h$. The appendix justifies this choice of boundary condition, which says that experts fully delever as their wealth shrinks.[16] Then, ODE (15) is solved on the endogenous region

[15]This also allows us to maintain consistency with Definition 2, which says that a W-BSE should have non-zero loading on the second shock.

[16]We use the boundary condition $\kappa(0) = 0$ in accordance with the literature. In Online Appendix D.1, we show that this is not necessary in principle. There are actually a continuum of W-BSEs indexed by $\kappa_0 = \kappa(0) \in [0, 1]$, which one can think of as agents' "disaster belief", i.e., what happens in the worst-case scenario. Nevertheless, there are good reasons to select $\kappa_0 = 0$. First, as we show in Online Appendix D.2, if managing capital involves any amount of idiosyncratic risk, even if vanishingly-small, any equilibrium must feature $\kappa \to 0$ as $\eta \to 0$. Second, Online Appendix D.3 shows how adding any amount of limited commitment frictions, even if vanishingly-small, automatically restricts equilibrium to feature $\kappa \to 0$ as $\eta \to 0$. Despite this discussion, several numerical examples in the paper use κ_0 slightly above zero, to

$(0, \eta^*)$ where households manage some capital, i.e., $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$.[17] Given a solution for (q, κ), the risk balance equation (RB) yields capital price variance as

$$\sigma_q^2 = \frac{\eta(1-\eta)}{\kappa - \eta} \frac{a_e - a_h}{q}, \quad \text{if} \quad \kappa < 1. \tag{16}$$

Since $\sigma_q \neq 0$ in (16), a W-BSE exists as long as a solution exists to ODE (15).

Proposition 1 (W-BSE). *If $\sigma = 0$, there exists a W-BSE with $\kappa(0) = 0$, in which $\sigma_q(\eta) \neq 0$ on $(0, \eta^*)$ and $\sigma_q(\eta) = 0$ on $(\eta^*, 1)$.*

Figure 2 displays a numerical example with the capital price q and volatility σ_q as functions of the expert wealth share. Notice that the equilibrium is stationary (the right panel of Figure 2 plots the stationary CDF of η).[18]

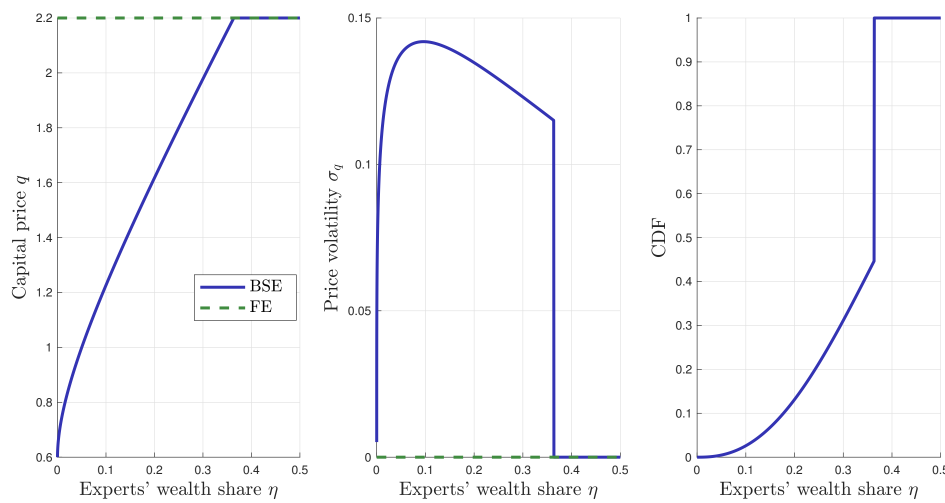

Figure 2: Capital price q, volatility σ_q, and stationary CDF of η. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$. OLG parameters (for the CDF): $\nu = 0.1$ and $\delta = 0.04$.

The intuition communicated by the W-BSE equations above is as follows. If agents believe the sunspot shock can affect asset prices, then the actual arrival of such a shock triggers trading of capital between experts and households. Since experts are more

aid numerical stability. This approximation is not problematic, given we show in Proposition D.1 that equilibria are continuous in κ_0, even at $\kappa_0 = 0$.

[17] When $\rho_h = \rho_e$, there is a closed form solution for capital price

$$q(\eta) = \frac{1}{\rho}\left[(a_e - a_h)\eta + a_h + \sqrt{((a_e - a_h)\eta + a_h)^2 - a_h^2}\right], \quad \text{for} \quad \eta < \eta^* = \frac{1}{2}\frac{a_e - a_h}{a_e}.$$

[18] This economy possesses a stationary distribution on $(0, \eta^*]$ under mild parameter restrictions, for example if experts are more impatient than households ($\rho_e > \rho_h$) or the economy has an OLG structure with sufficiently few experts ($\delta > 0$ and $\nu < \eta^*$). Note that $\eta_t = \eta^*$ about 55% of the time in the numerical example of Figure 2, i.e., there is a mass point at η^*.

productive than households, capital transfers have real effects and move asset prices. But it does not end there: asset-price fluctuations feed back into the wealth distribution, which initiates another round of capital transfers, and so on. The question "does there exist an initial belief about asset prices that can be self-justified by this process?" is tantamount to solving the ODE (15).

2.2 W-BSEs are inconsistent with fundamental shocks

The previous section shows how a W-BSE can arise without fundamental shocks ($\sigma = 0$). But with fundamental shocks ($\sigma \neq 0$), we obtain the stark result that, in an equilibrium that is Markov in η, capital prices must be completely insensitive to the sunspot shock $Z^{(2)}$. In this sense, W-BSEs are not robust to the inclusion of fundamental uncertainty.

To see this, solve for the shock loadings σ_η and σ_q. Following the same analysis leading to equation (14), we obtain an equation for σ_q:

$$\Big[1 - (\kappa - \eta)\frac{q'}{q}\Big]\sigma_q = \big(\begin{smallmatrix}1\\0\end{smallmatrix}\big)(\kappa - \eta)\sigma\frac{q'}{q}. \tag{17}$$

Equation (17) is really two equations stacked. Given $\sigma \neq 0$, the first equation can only hold if $(\kappa - \eta)\frac{q'}{q} \neq 1$. This is inconsistent with the second equation, unless $\sigma_q \cdot \big(\begin{smallmatrix}0\\1\end{smallmatrix}\big) = 0$. Thus, we have proved

Lemma 2. *If $\sigma \neq 0$, any Markov equilibrium in η is insensitive to sunspot shocks.*

2.3 The W-BSE is approximately a fundamental equilibrium

Lemma 2 shows that, in the presence of fundamental shocks, a Markovian equilibrium in η must be a Fundamental Equilibrium (FE). These FE are studied extensively in the literature, with the defining feature that fundamental shocks are amplified by endogenous wealth dynamics (Brunnermeier and Sannikov, 2014). We analyze and discuss these FE in Online Appendix E.[19]

To briefly recap these FE, rearrange equation (17) to obtain

$$\sigma_q = \frac{(\kappa - \eta)q'/q}{1 - (\kappa - \eta)q'/q}\sigma. \tag{18}$$

[19] As a new but tangential result, this online appendix also demonstrates the multiplicity of FEs along two dimensions, κ_0 and $\mathrm{sgn}(\sigma_R)$, neither of which have been documented in the literature.

Equation (18) is often interpreted as *amplification*, because $\frac{(\kappa-\eta)q'/q}{1-(\kappa-\eta)q'/q}$ takes the form of a convergent geometric series. In words, a negative fundamental shock reduces experts' wealth share η directly through $(\kappa-\eta)\sigma$, which reduces asset prices through q'/q. This explains the numerator of (18). But the reduction in asset prices has an indirect effect: a one percent drop in capital prices reduces experts' wealth share by $(\kappa-\eta)$, which feeds back into a $(\kappa-\eta)q'/q$ percent further reduction capital prices, which then triggers the loop again. The second-round impact is $[(\kappa-\eta)q'/q]^2$, and so on. This infinite series is convergent if $(\kappa-\eta)q'/q < 1$, such that incremental amplification is reduced in each successive round of the feedback loop.

In the W-BSE, recall that $(\kappa-\eta)q'/q = 1$ (equation (14)). This BSE has no dampening in successive rounds of the feedback loop, leading to infinite amplification!

Despite this contrast, it turns out that the W-BSE is "close" to these FE. As σ shrinks, amplification rises because falling exogenous volatility incentivizes expert leverage, which raises endogenous volatility. As σ vanishes, amplification rises explosively and equilibria become sunspot-like.[20]

Lemma 3. *Suppose a Markov equilibrium in η exists for each $\sigma > 0$ small enough, with $\kappa(0) = 0$. As $\sigma \to 0$, the equilibrium converges to the W-BSE.*

Thus, even if fundamentals are truly deterministic, our W-BSE "looks similar" to the FEs that have been studied in the literature. This approximate observational equivalence implies the W-BSE cannot possibly generate the type of novel dynamics promised in the introduction.

3 Beyond wealth: sentiment-driven equilibria

Section 2 says that Markov equilibria in experts' wealth share η are either (a) pure FE (Lemma 2); or (b) look very much like pure FE (Lemma 3). To address this critique, we endeavor here to analyze a richer class of BSEs that are not Markov in η. Below, we establish some sufficient conditions for existence of such equilibria, and then we provide detailed characterization of these equilibria.

Because the capital price q is the critical endogenous object (one may think of q as the "co-state" variable), equilibria which are not Markov in η share the defining characteristic that a variety of different asset prices can prevail for a given wealth distribution. Since

[20] Brunnermeier and Sannikov (2014) provide a related limiting result, arguing that asset-price volatility does not vanish as $\sigma \to 0$, also known as the "volatility paradox." Related results can be found in Manuelli and Peck (1992) and Bacchetta et al. (2012), in which sunspot equilibria could be seen as limits of fundamental equilibria when fundamental uncertainty vanishes.

η captures all fundamental information in this economy, one can think of "sentiment" as responsible for generating the multiplicity of asset prices corresponding to the same η. This is why Definition 2 refers to this class of equilibria as Sentiment-driven BSEs, which we restate here for convenience.

Definition 3. A Sentiment-driven BSE (S-BSE) is a BSE that is not Markov in η.

Remark 2 (Stability and multiplicity: connection to literature)**.** *Stability is the critical property enabling sunspots in deterministic dynamical systems. For example, recall the neoclassical growth model, in which capital and consumption are the state and co-state variables, respectively, and only one value of initial consumption is consistent with a non-explosive equilibrium. By contrast, OLG versions of the growth model can feature a stable steady state, to which many alternative values of initial consumption would converge (Azariadis, 1981; Cass and Shell, 1983). This literature generates stochastic sunspot equilibria by basically randomizing over the multiplicity of transition paths.*

S-BSEs will also feature a type of stability, whereby for a fixed initial wealth distribution η_0, many initial values of the co-state q_0 can be consistent with non-explosive behavior. But the analogy to deterministic models breaks down in an important sense: Online Appendix D.2 shows that the deterministic steady state of our class of models is only saddle-path stable, so we cannot *obtain volatility by randomizing over a multiplicity of deterministic transition paths. For the same reason, we cannot hard-wire arbitrary amounts of volatility for any combination (η, q). Rather, as will soon be clear, our model uniquely determines return volatility $|\sigma_R|$ for each (η, q), reminiscent of the endogenously-determined sentiment distribution in Benhabib et al. (2015).*

We make some mild parameter restrictions and then present the main results.

Assumption 1. *Parameters satisfy (i) $0 < \frac{a_h}{\rho_h} < \frac{a_e}{\rho_e} < +\infty$; (ii) $\sigma^2 < \rho_e(1 - a_h/a_e)$; and (iii) either $0 < \delta\nu < \delta$, or $\sigma^2 < \rho_e - \rho_h$.*

Assumption 1 part (i), only for convenience, makes the very modest assumption that the capital price is higher if experts control 100% of wealth than if households control 100% of wealth. Part (ii), meant to make the problem interesting, ensures experts sometimes hold all capital (i.e., $\kappa = 1$) and sometimes do not ($\kappa < 1$). Part (iii) guarantees experts do not asymptotically hold all wealth.

Theorem 1 (Existence of S-BSEs)**.** *Let Assumption 1 hold. Then, there exists an S-BSE in which $(\eta_t, q_t)_{t \geq 0}$ remains in $\mathcal{D} := \{(\eta, q) : 0 < \eta < 1 \text{ and } \eta a_e + (1 - \eta)a_h < q\bar{\rho}(\eta) \leq a_e\}$ almost-surely and possesses a non-degenerate stationary distribution.*

Theorem 1 is proved in Appendix B.1 with an explicit S-BSE construction. To understand its properties and the challenges in the construction, we first explain the static mechanism that allows sunspot volatility and then the dynamic mechanism that prevents sunspot volatility from becoming explosive.

Static indeterminacy mechanism. Given a wealth distribution η and a level of return volatility $|\sigma_R|$, the capital market is equilibrated at each time via the risk-balance condition (RB) and the price-output relation (PO), restated here for convenience:

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}|\sigma_R|^2\right] \tag{RB}$$

$$q\bar{\rho} = \kappa a_e + (1-\kappa)a_h. \tag{PO}$$

The left panel of Figure 3 shows how the intersection of these two curves determines the capital allocation κ and the capital price q. The downward-sloping risk-balance (RB) can be thought of as experts' relative capital demand: for a fixed level of wealth η and return volatility $|\sigma_R|$, experts will only hold more capital if it is cheaper, thereby offering a higher expected return. The upward-sloping price-output (PO) is a capital supply curve: experts' capital provision raises allocative efficiency and capital valuations.

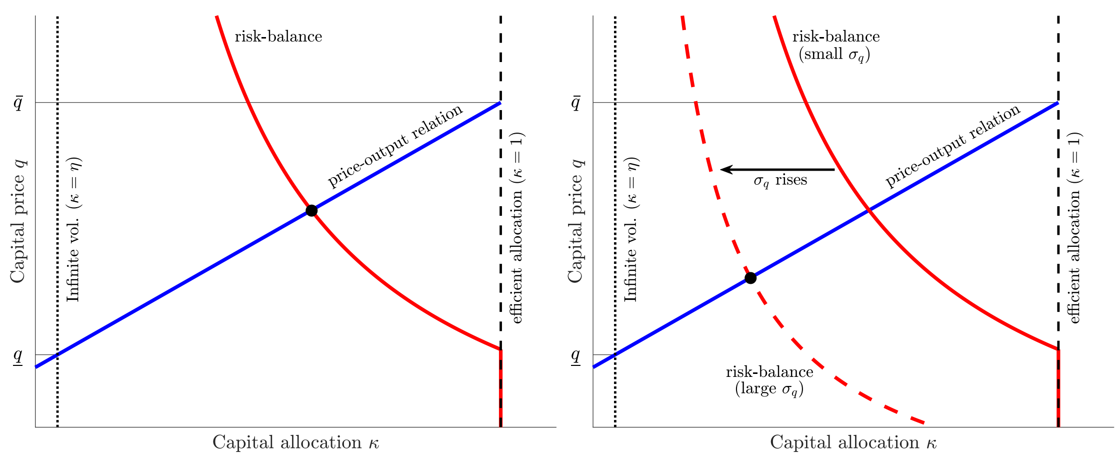

Figure 3: Static sunspot mechanism. Both panels plot the risk-balance condition (RB) and price-output relation (PO) for a fixed level of $\eta = 0.2$. The horizontal lines labeled $\bar{q}$ and $\underline{q}$ refer to maximal and minimal possible values of the capital price, respectively, corresponding to an efficient capital allocation ($\kappa = 1$) and an infinite-volatility allocation ($\kappa = \eta$). *Left panel:* equilibrium with $|\sigma_R| = 0.13$. *Right panel:* equilibrium after a shift to $|\sigma_R| = 0.20$. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, and $\sigma = 0.10$.

But whereas η is a state variable that can be rightly treated as fixed in this static sense, return volatility $|\sigma_R|$ is not. The right panel of Figure 3 shows what changes if there is a sudden rise in *fear*, manifested as higher perceived volatility $|\sigma_R|$. Experts, being risk-averse, are less willing to hold capital when volatility is high. This is illustrated as a leftward shift in the risk-balance curve from the solid to the dashed line. The new allocation, after this "fire sale," features a less efficient capital allocation, lower asset prices, and higher volatility.

So far, nothing rules out this arbitrary rise in fear, and $|\sigma_R|$ appears indeterminate. The indeterminacy in $|\sigma_R|$ translates into an indeterminacy in q, which can be seen by combining (RB) and (PO) to obtain the negative price-variance association:

$$|\sigma_R|^2 = \frac{\eta(1-\eta)(a_e - a_h)^2}{q\bar{\rho}(\eta) - \eta a_e - (1-\eta)a_h}\frac{1}{q}, \quad \text{when } \kappa < 1. \tag{19}$$

Therefore, static restrictions are consistent with many solutions for q_0, given any η_0, as required for an S-BSE. Each q_0 corresponds to a different σ_R, by (19).

Remark 3 (Price-output)**.** *Our multiplicity rests upon a link between asset prices and output efficiency. Without a price-output link, capital ownership cannot affect prices (e.g., if $a_e = a_h = a$, then $q = a/\bar{\rho}$ independently of κ).*[21] *The role of financial frictions is only to create a non-trivial price-output link: with complete markets, risk-sharing decouples from capital ownership, and experts always manage all capital. This centrality of the price-output link is clarified further in Khorrami and Mendo (2021), which presents a complete-markets New Keynesian economy that supports self-fulfilling fluctuations because a link between asset prices and output efficiency arises at the zero lower bound.*

Dynamic stability mechanism. The set of prices q_0 supported by the static indeterminacy above will only be an equilibrium if it does not lead to explosive paths. This is a real concern here: with higher volatility in the new candidate equilibrium, any subsequent fear shocks would have a larger direct impact, further raise volatility, and so on, ad infinitum. Formally, as $\kappa \to \eta$, we have $|\sigma_R| \to +\infty$. Intuitively, volatility must be explosive because agents with heterogeneous productivities but identical preferences will take identical portfolio positions only if risk is so enormous that it swamps other considerations.

With unbounded volatility, there is an imminent violation of the equilibrium conditions. For example, κ would fall below η, and (RB) could not hold. Rational forward-looking agents would rule this out at the beginning, decide to suppress their fear, and

[21] Our model, like Bacchetta et al. (2012), always possesses a negative relationship between asset prices and volatility—see equation (19). Nevertheless, general equilibrium imposes strong discipline: from (PO), asset prices are given by $a_e/\bar{\rho}$ in the cases $a_h = -\infty$ or $a_e = a_h$. This discipline occurs through endogenous adjustments by the interest rate. Benhabib et al. (2020) show that "self-fulfilling risk panics" of Bacchetta et al. (2012) require asset prices to have a direct impact on the stochastic discount factor, which is exactly what happens with a price-output link. (In Benhabib et al. (2020), the price-SDF link can arise either due to OLG or collateral constraints.)

there would be no indeterminacy. Thus, the static indeterminacy mechanism is incomplete without some force that also prevents this type of explosion. That force is the drift μ_q, which is indeterminate and can be judiciously chosen to keep the dynamical system $(\eta_t, q_t)_{t\geq 0}$ stable.

The key observation is that optimal capital holdings are a function of the *risk premium*. This is clearly visible in the optimal portfolio FOCs (6)-(7), where only the spread $\mu_q - r$ appears. Consequently, only the spread $\mu_q - r$ is pinned down in equilibrium, as equation (9) shows; μ_q and r are not separately determined.

Making a judicious choice for μ_q is straightforward. Because $(\eta_t, q_t)_{t\geq 0}$ evolves in a diffusive fashion, stability criteria conveniently boil down to boundary behavior of the dynamical system. Thus, by imposing certain boundary conditions on μ_q, we prevent explosive volatility and ensure a stochastically stable system. For example, we can impose that $\mu_q \to +\infty$ if q falls too low, and $\mu_q \to -\infty$ if q rises too high.

Such a choice of μ_q represents a belief that the economy will "bounce back" from extreme states. A priori, it is hard to say whether such a belief is reasonable or not, although it is required for our self-fulfilling mechanism. What we can say is that such beliefs are a relatively minimal requirement, given the vanishingly small probability of approaching these extreme states.

With these bounce-back beliefs, an entire fear-driven sequence of asset price drops can be justified. Intuitively, agents understand that future capital price dynamics will keep things stationary and prevent explosive behavior, so current prices q_0 can take essentially arbitrary values.

Corollary 1 (Price and volatility indeterminacy)**.** *Given initial wealth share $\eta_0 \in (0,1)$, let $\mathcal{Q}(\eta_0)$ denote the set of possible S-BSE values of q_0, and let $\mathcal{V}(\eta_0)$ denote the associated set of possible S-BSE values of return variance $|\sigma_R(\eta_0, q_0)|^2$. Then,*

$$\mathcal{Q}(\eta) = \begin{cases} \left(\frac{\eta a_e + (1-\eta)a_h}{\bar{\rho}(\eta)}, \frac{a_e}{\bar{\rho}(\eta)}\right), & \text{if } \eta < \eta^* := \frac{\rho_h}{\rho_e}\left(\frac{1 - a_h/a_e}{\sigma^2}\rho_e - 1 + \frac{\rho_h}{\rho_e}\right)^{-1}; \\ \left(\frac{\eta a_e + (1-\eta)a_h}{\bar{\rho}(\eta)}, \frac{a_e}{\bar{\rho}(\eta)}\right], & \text{if } \eta \geq \eta^*, \end{cases}$$

and

$$\mathcal{V}(\eta) = \Big(\min\left[\eta\bar{\rho}(\eta)\frac{a_e - a_h}{a_e}, \sigma^2(\bar{\rho}(\eta)/\rho_e)^2\right], +\infty\Big).$$

In particular, if fundamental volatility $\sigma = 0$, then return variance spans any value between 0 and $+\infty$, regardless of the wealth distribution η. Finally, an S-BSE can be constructed such that, in the stationary distribution, positive probability is placed on all elements of $\mathcal{Q}(\eta)$ and $\mathcal{V}(\eta)$.

Figure 4 plots the admissible set of η and q, along with return volatility $|\sigma_R|$ (indicated by shading) at each point in the space. For reference, we also place the W-BSE (which has $\sigma = 0$) and a Fundamental Equilibrium (with $\sigma = 0.1$). These equilibria attain only 10-20% volatility, a tiny amount of what S-BSEs can do.

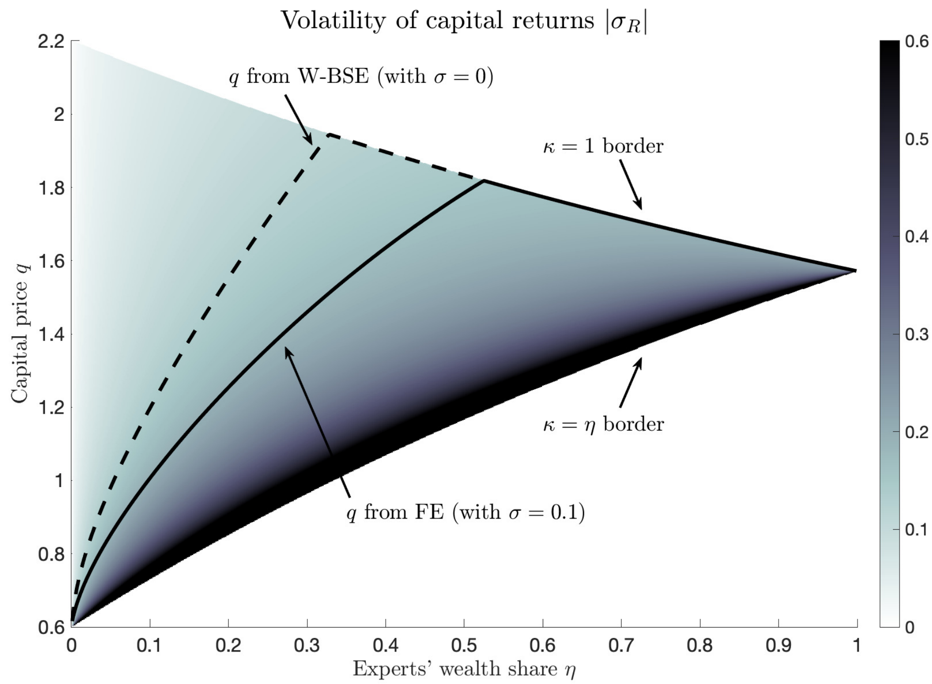

Figure 4: Colormap of volatility $|\sigma_R|$ as a function of (η, q), in the region $\mathcal{D} := \{(\eta, q) : \eta \in (0,1)$ and $\eta a_e + (1-\eta)a_h < q\bar{\rho}(\eta) \leq a_e\}$. Volatility is truncated for aesthetic purposes (because $|\sigma_R| \to \infty$ as $\kappa \to \eta$). For reference, also included are the W-BSE with $\sigma = 0$ and the Fundamental Equilibrium (FE) with $\sigma = 0.1$. Parameters: $\rho_e = 0.07$, $\rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$.

Two important indeterminacies in S-BSEs. Besides the vast amount of multiplicity in q_0, S-BSEs also allow: (1) arbitrary decoupling of volatility from fundamentals; and (2) almost any degree of persistence or transience. We formalize these statements, which follow directly from the construction in Theorem 1, and then discuss intuition.

Corollary 2 (Decoupling)**.** *The economy can be arbitrarily coupled or decoupled from fundamentals in the following sense. Let $\gamma(\eta, q) \in [0,1]$ be any C^1 function. An equilibrium exists such that when $\kappa < 1$, a fraction $\gamma(\eta, q)$ of return variance $|\sigma_R|^2$ is due to the fundamental shock.*

S-BSEs do not pin down the fraction of volatility stemming from the fundamental and sunspot shocks, $Z^{(1)}$ and $Z^{(2)}$, respectively. The reason: when trading, agents only care about total return variance, not its source. Mathematically, the price-variance association (19) is a single equation relating q and $|\sigma_R|$, but σ_R itself has two components that can make indeterminate contributions to equilibrium. Consequently, asset prices and economic activity can be either closely linked to fundamentals, or completely decoupled from them, and this decoupling can be time-varying in arbitrary ways. Nevertheless, the next section presents perhaps the most natural example of an S-BSE, in which volatility and fundamentals must decouple as total volatility rises.[22]

Corollary 3 (Drift indeterminacy)**.** *The economy can feature any degree of persistence or transience in the following sense. Let $m(\eta, q)$ be any C^1 function. An equilibrium exists with $\mathbb{P}[\mu_{q,t} = m(\eta_t, q_t) \mid \kappa_t < 1]$ arbitrarily close to one. Furthermore, the inefficiency probability $\mathbb{P}[\kappa_t < 1]$ can take any value between zero and one.*

As suggested earlier, the proof of Theorem 1 only imposes certain boundary conditions on μ_q, which allows almost any behavior in the interior of the state space. For example, asset prices could almost always behave like a random walk (corresponding to $\mu_q \approx 0$ in the interior), with just enough mean-reversion in extreme states to keep things stationary; in such a design, extreme states become arbitrarily close to reflecting boundaries. Alternatively, fluctuations could be much more transitory in nature. In the next section, we harness the indeterminacy in μ_q to address predictability of busts and speed of recovery.

4 Resolving puzzles with sentiment

We have just demonstrated that sunspot equilibria, which are endemic to this class of models, in principle can support rich dynamics. Now, we solve some concrete examples to illustrate several substantive results along these lines.

4.1 Explicit construction with a sentiment state variable

In contrast to the previous subsection's non-Markovian setting (where q acted as the co-state variable), here we implement our sunspot equilibria with an explicit state variable. These equilibria are essentially special cases of the S-BSEs in Section 3, but being explicit about a sentiment state variable is useful for several reasons. First, the Markov equilibrium construction will be pedagogically more familiar to the literature on sunspots. Second, adding a sentiment state variable brings some clarity, as the sentiment state dynamics can be modeled as locally uncorrelated with fundamental shocks. Third, this

[22]We implement the example next section with a sentiment state variable s_t whose innovations depend only on the sunspot shock $Z^{(2)}$, which we think of as a natural case. More generally, letting ϕ be the (fixed) correlation between ds and $dZ^{(1)}$, one can show that as volatility rises ($|\sigma_R| \to \infty$), the fraction of return variance attributable to the fundamental shock approaches ϕ^2.

setting happens to facilitate building sunspot equilibria in which experts fully de-lever as their wealth shrinks, i.e., $\kappa \to 0$ as $\eta \to 0$, for which there are natural justifications.

Let s be a pure sunspot that is irrelevant to economic fundamentals and loads on only the second shock (recall $Z^{(1)}$ affects capital and $Z^{(2)}$ does not):[23]

$$ds_t = \mu_{s,t}dt + \sigma_{s,t}\binom{0}{1} \cdot dZ_t, \quad s_t \in \mathcal{S}. \tag{20}$$

The time-varying drift and diffusion of s capture cleanly the non-iid nature of sentiment shocks. By contrast, Section 2 did not include additional state variables governing sentiment dynamics, effectively restricting sentiment shocks to be iid. This is one way to view the core distinction between the W-BSEs of Section 2 from the S-BSEs of Section 3.

We will also find some use in introducing auxiliary state variables that can (only) affect the drift $\mu_{s,t}$. This is possible to do in a very flexible way, due to the drift indeterminacy result of Corollary 3. Let $x_t \in \mathcal{X}$ be an arbitrary bounded diffusion (perhaps allowing reflections at the boundaries of the state space $\mathcal{X}$),

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t) \cdot dZ_t,$$

which affects the sentiment drift, through $\mu_{s,t} = \mu_s(\eta_t, s_t, x_t)$.

Definition 4. A *Markov S-BSE* in states $(\eta, s, x) \in (0,1) \times \mathcal{S} \times \mathcal{X}$ consists of functions $(q, \kappa, r, \sigma_\eta, \mu_\eta, \sigma_s) : (0,1) \times \mathcal{S} \mapsto \mathbb{R}$, and $\mu_s : (0,1) \times \mathcal{S} \times \mathcal{X} \mapsto \mathbb{R}$, all C^2 almost-everywhere, such that the process $(\eta_t, q(\eta_t, s_t), \kappa(\eta_t, s_t), r(\eta_t, s_t))_{t \geq 0}$ is an S-BSE.

Remark 4 (Endogenous sentiment dynamics)**.** *Note that the statement of Definition 4 allows (σ_s, μ_s) to be endogenous, in the sense that they could depend on the wealth distribution η. Our examples in this section purposefully entertain this endogeneity, partly because we think of this as the more interesting and realistic situation. Why? As shown in Section 3, dynamics depend explicitly on q in an S-BSE. Thus, it is completely sensible for agents in our S-BSEs to use asset prices directly in forecasting; in particular, sentiment dynamics (σ_s, μ_s)—which are nothing but belief dynamics—themselves should condition on q. But q will depend on both s and η, implying sentiment dynamics (σ_s, μ_s) depend on η too, through q. That said, Online Appendix D.6 verifies that similar types of sunspot equilibria can be constructed with exogenous sentiment dynamics, i.e., (σ_s, μ_s) are only functions of s, not η.*[24]

[23]This is a natural choice that also brings clarity. We have also solved examples with sentiment correlated to fundamentals, i.e., with $ds = \mu_s dt + \sigma_s^{(1)} dZ^{(1)} + \sigma_s^{(2)} dZ^{(2)}$. An additional feature that emerges relative to what we show here is that $\sigma_s^{(1)}$ can work to reduce asset price volatility at times, unlike $\sigma_s^{(2)}$. See Online Appendix D.5 for details.

[24]Because it may be more natural to think that agents coordinate on variables that have direct real ef-

The equilibrium conditions are derived similarly to previous sections. By applying Itô's formula to $q(\eta, s)$, we obtain the capital price volatility σ_q in terms of σ_η. From equation (12), we also have σ_η in terms of σ_q. Solving this two-way feedback, we obtain

$$\sigma_q = \frac{\binom{1}{0}(\kappa - \eta)\sigma\partial_\eta \log q + \binom{0}{1}\sigma_s\partial_s \log q}{1 - (\kappa - \eta)\partial_\eta \log q}. \tag{21}$$

Using (21) in (RB), we obtain the following equation linking capital prices, the capital distribution, and sentiment volatility:

$$0 = \min\left[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}\left(\frac{\sigma^2 + (\sigma_s\partial_s \log q)^2}{(1 - (\kappa - \eta)\partial_\eta \log q)^2}\right)\right]. \tag{22}$$

Our strategy to find a Markov S-BSE is to guess a capital price function $q(\eta, s)$ and then use equation (22) to "back out" the sunspot volatility σ_s that justifies it. We purposefully perform this construction such that sunspots only increase volatility, to highlight their potential for resolving puzzles.

More specifically, suppose a fundamental equilibrium, where sunspots do not matter, exists with equilibrium capital price q^{FE} (see Online Appendix E for details on the fundamental equilibria). We will think of q^{FE} as the "best-case" capital price, because despite featuring amplification, q^{FE} inherits no sunspot volatility. Conversely, think of the capital price q^∞ associated to an infinite-volatility equilibrium as the "worst-case" capital price (substitute $|\sigma_R| \to \infty$ into (19) to find $q^\infty := \frac{\eta a_e + (1-\eta)a_h}{\bar{\rho}}$).

Our strategy is essentially to treat the sentiment variable s as a device to shift continuously between the best-case q^{FE} and the worst-case q^∞. Mathematically, we conjecture a capital price that is approximately a weighted average of q^{FE} and q^∞, with weights s and $1 - s$.[25] The novelty of our approach here is to then use equation (22) to solve for sunspot volatility σ_s, which will generically depend on experts' wealth share η. In terms of Figure 4, the economy will live in the sub-region bounded by the solid FE line and

fects, Online Appendix D.6 also entertains the situation that s impacts fundamental volatility σ. In this environment, we illustrate two main results. First, depending on coordination on s, equilibrium can either behave more "intuitively" or less. Intuition usually says that prices move inversely to fundamental volatility, but coordination can easily reverse this relationship. Second, as the contribution of s to fundamental volatility becomes small, equilibrium can converge to one where s still matters, i.e., a sunspot equilibrium. Thus, sunspot equilibria are limits of fundamental equilibria, with a vanishingly-small real contribution of s.

[25] In this particular equilibrium, capital prices can never literally achieve the "worst-case" capital price q^∞, for two technical reasons, both of which ensure that sunspot volatility stays σ_s bounded: (i) to ensure $\kappa(0, s) = 0$, we need $q(\eta, s)$ to behave like the fundamental solution $q^{FE}(\eta)$ for η close enough to zero, and all s; (ii) we need $q(\eta, s) > q^\infty(\eta)$, so that $\kappa(\eta, s) > \eta$ for all (η, s). Thus, in the proof of Proposition 2, we actually construct q^∞ as a close approximation to the worst-case price, such that (i) and (ii) are satisfied.

the $\kappa = \eta$ border (and notice this implies that the full-deleveraging condition $\kappa \to 0$ as $\eta \to 0$ thus holds). In the proposition below, we verify that such a construction is indeed an equilibrium.

Proposition 2. *Let Assumption 1 hold, and assume a fundamental equilibrium exists for each $\sigma \geq 0$ small enough. Then, for all $\sigma \geq 0$ small enough, there exists a Markov S-BSE with capital prices arbitrarily close to $sq^{FE}(\eta) + (1-s)q^{\infty}(\eta)$. In this equilibrium, μ_s is indeterminate except near the boundaries of $(0,1) \times \mathcal{X} \times \mathcal{S}$.*

We construct a numerical example closely following Proposition 2, which we will use in subsequent sections. The left panel of Figure 5 shows the capital price function. Positive sunspot shocks reduce the capital price, independently of wealth share η (although η will also endogenously respond to s-shocks).

The middle panel of Figure 5 displays capital return volatility, which can be substantially greater than in the fundamental equilibrium. Implied by capital return volatility is an underlying sunspot shock size σ_s, which is displayed in the right panel of Figure 5. Sunspot dynamics become more volatile both as experts become poor (η shrinks) and as the economy approaches the worst-case equilibrium (s rises). The dependence of σ_s on η is the notion of endogenous beliefs that can occur in S-BSEs.

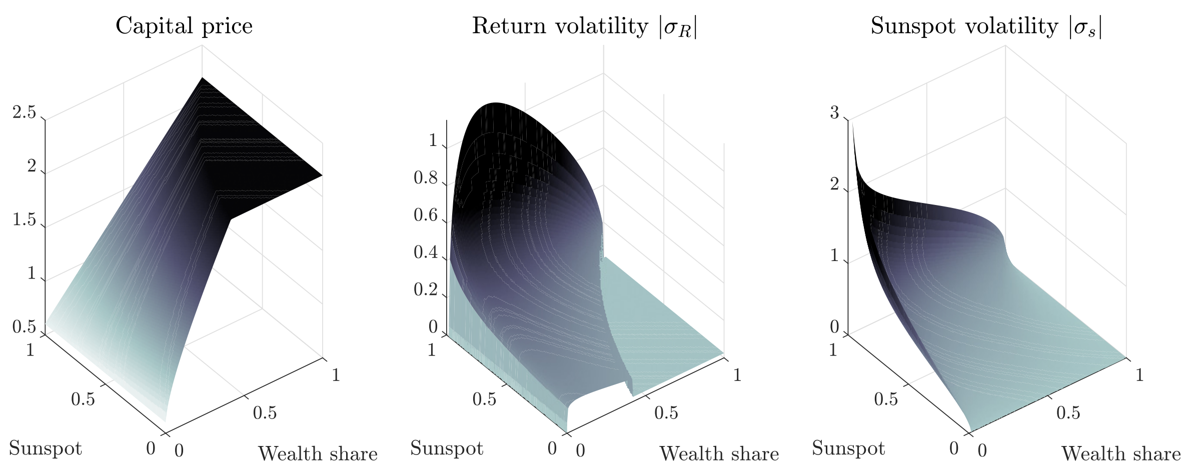

Figure 5: Capital price q, volatility of capital returns $|\sigma_R|$, and sunspot shock volatility σ_s. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$.

4.2 Non-fundamental crises and large amplification

We now show hour our model with sentiment shocks naturally resolves some empirical issues related to financial crises and recoveries.

First, Figure 6 compares impulse responses to a large negative balance-sheet shock (i.e., decline in η) versus a sunspot (i.e., increase in s). The shock sizes are chosen so that the initial drop in capital price $q_0 - q_{0-}$ is roughly the same. "Balance-sheet recessions" (decline in η) feature a modest increase in volatility followed by relatively slow recoveries, as experts can only rebuild their balance sheets by earning profits over time. By contrast, "self-fulfilled crises" (increase in s) feature large temporary volatility spikes and can have accelerated recoveries (depending on the choice of μ_s). The dynamics after a sentiment shock—both the rise in volatility and speed of recovery—are closer to empirical evidence.[26] Our results on recovery speeds are related to Maxted (2020), who shows how extrapolative beliefs can help this class of models match such evidence, but with our rational sentiment in place of his behavioral sentiment.

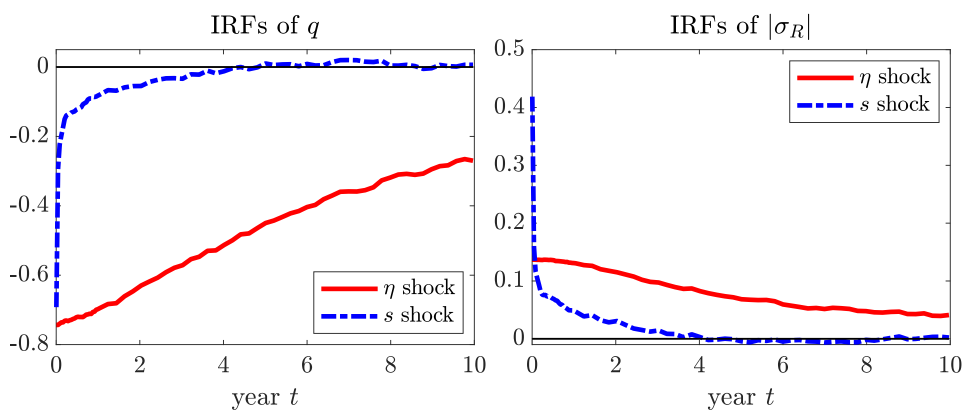

Figure 6: Bust IRFs of capital price q and return volatility $|\sigma_R|$. The IRFs labeled "η shock" are responses to a decrease in η from $\eta_{0-} = 0.5$ to $\eta_0 = 0.2$, holding s_0 fixed at 0.1. The IRFs labeled "s shock" are responses to an increase in s from $s_{0-} = 0.1$ to $s_0 = 0.9$, holding η_0 fixed at 0.5. These shock sizes are chosen such that the initial response of q are approximately equal. Note that η_0 would respond to an "s shock," since σ_η has a non-zero second element, but we keep it fixed here. IRFs are computed as averages across 500 simulations at daily frequency. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$. In this example, we set the sunspot drift $\mu_s = 0.0002s^{-1.5} - 0.0002(s_{\max} - s)^{-1.5}$, where $s_{\max} = 0.95$. This choice ensures $s_t \in (0, s_{\max})$ with probability 1.

To establish some more confidence in these results, we present the following two propositions which together show that amplification can be arbitrarily high (Proposition 3) as long as sentiment shocks are the source (Proposition 4). Given the literature's struggle to identify a "smoking gun" (e.g., TFP shocks, capital efficiency shocks) for financial

[26] During the 2008 financial crisis and 2020 COVID-19 episode in the US, implied volatility from option markets spiked by magnitudes on the order of 60%. For a rough idea of what the data says about crisis recoveries, see Jordà et al. (2013) and Reinhart and Rogoff (2014) for output, and see Muir (2017) and Krishnamurthy and Muir (2017) for credit spreads and stock prices. Across these many measures, and using broad-based international panels, crisis recovery times tend to range from 4-6 years on average.

Of course, note that η responds to s-shocks, i.e., σ_η has a non-zero second component. Thus, a true sentiment-driven crisis features dynamics that are a blend of the two IRFs in Figure 6. Figure 6 shows a pure shock to s, without the endogenous co-movement in η, for theoretical clarity.

crises, we view this set of results as a helpful insight. The importance of sentiment s, relative to experts' wealth share η, also echoes the empirical results suggesting financial crises are not associated with pre-crisis levels of bank capital (Jordà et al., 2021).

Proposition 3 (Arbitrary volatility)**.** *Given a target variance $\Sigma^* > 0$ and any parameters satisfying the assumptions of Proposition 2, there exists a Markov S-BSE with stationary average return variance exceeding the target, i.e., $\mathbb{E}[|\sigma_R|^2] > \Sigma^*$.*

Proposition 4 (Volatility decoupling)**.** *In the Markov S-BSEs of Proposition 2, both the fraction of return volatility due to sentiments $|\binom{0}{1} \cdot \sigma_R| / |\sigma_R|$ and total return volatility $|\sigma_R|$ increase with s.*

4.3 Booms predict crises

We now use the same framework to cast light on empirical findings suggesting that financial crises are predictable, in particular by large credit and asset price booms (Reinhart and Rogoff, 2009; Jordà et al., 2011, 2013, 2015a,b; Mian et al., 2017) that feature below-average credit spreads (Krishnamurthy and Muir, 2017; López-Salido et al., 2017; Baron and Xiong, 2017).

To do this, we make use of the auxiliary variable x that can affect the sentiment drift. Following some models of extrapolative beliefs (Barberis et al., 2015; Maxted, 2020), define an exponentially-declining weighted average of sentiment shocks:

$$x_t := x_0 + \sigma_x \int_0^t e^{-\beta_x(t-u)} dZ_u^{(2)}. \tag{23}$$

Assume the drift of s depends on x via

$$\mu_{s,t} = b_x x_t + \hat{\mu}_s(s_t) \quad \text{with} \quad b_x \leq 0.$$

Similar to Section 4.2, the term $\hat{\mu}_s$ will be designed to prevent non-stationarity in s_t. The new term $b_x x$ induces the following dynamics: after a series of good sentiment shocks ($dZ_t^{(2)} < 0$), s_t and x_t will be low, but this buoys $\mu_{s,t}$ and shifts conditional distributions of s_{t+h} to the right. If the constant b_x is large enough, the shift can generate dynamics reminiscent of "overshooting," in which a sentiment-driven boom raises bust probabilities. Differently from the extrapolative belief literature, the beliefs implied by these sentiment dynamics are completely rational.

Figure 7 displays IRFs consistent with this overshooting logic. A positive sentiment shock raises asset prices and lowers volatility for 2-3 years, but predicts mildly lower prices and higher volatility afterward. The right panel shows a tail risk measure, namely the probability that volatility exceeds 0.25 in the future, which emphasizes the overshooting logic is particularly salient for the extreme adverse outcomes. This is the sense in which a sentiment-driven boom has predictive power for a future bust and particularly a future crisis. By contrast, a boom driven by expert wealth counterfactually predicts high prices, lower volatility, and lower fragility at all horizons.

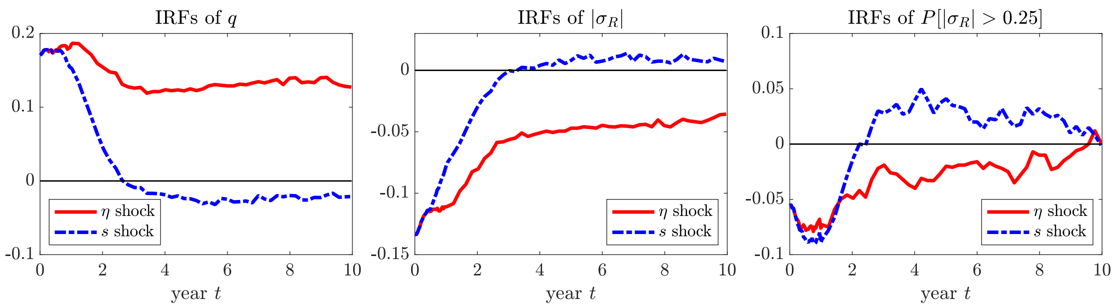

Figure 7: Boom IRFs of capital price q, return volatility $|\sigma_R|$, and the probability that $|\sigma_R| > 0.25$ at some point over future years. The IRFs labeled "η shock" are responses to an increase in η from $\eta_{0-} = 0.5$ to $\eta_0 = 0.7$, holding s_0 fixed at 0.4. The IRFs labeled "s shock" are responses to a decrease in s from $s_{0-} = 0.4$ to $s_0 = 0.1$, holding η_0 fixed at 0.5. These shock sizes are chosen such that the initial response of q are approximately equal. Note that η_0 would respond to an "s shock," since σ_η has a non-zero second element, but we keep it fixed here. IRFs are computed as averages across 1000 simulations at daily frequency. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$. In this example, we set the sunspot drift $\mu_s = b_x x + 0.0002 s^{-1.5} - 0.0002(s_{\max} - s)^{-1.5}$, where $s_{\max} = 0.95$, $b_x = -10$, $\beta_x = 0.1$, and $\sigma_x = 0.025$. The parameters (β_x, σ_x) are approximately the values used for the mean-reversion and volatility of the diagnostic belief process in Maxted (2020).

To connect more directly to the empirical literature, we conduct a financial crisis event study in Figure 8. We simulate our model (which thus features contributions from both fundamental and sunspot shocks), identify crises in the simulated data, and plot average outcomes in the several years before and after crisis. Crises are identified using the worst 3rd percentile of monthly output drops, but any other tail outcome will produce similar graphs. We see that conditions are improving up to 2 years before the crisis, with risk premia below average and *declining*. The crisis emerges suddenly and features spikes in all variables. Although we do not report it here, such dynamics cannot be produced in the non-sunspot equilibria of the model.

4.4 Sentiment-based jumps

In our final exercise, we show how similar substantive results—large and sudden crises that are preceded by booms featuring low volatility and risk premia—also hold in alternative equilibria with sentiment-based jumps. There are three reasons why jump-type fluctuations are an interesting avenue to explore vis á vis the puzzles in this literature.

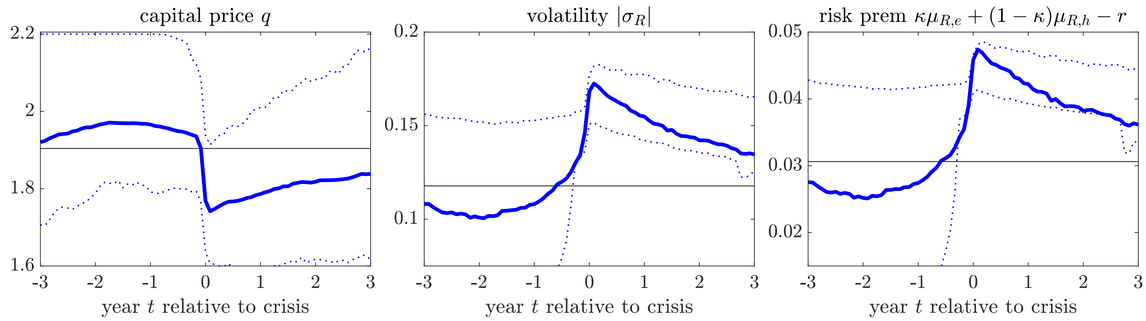

Figure 8: Event studies around financial crises. Crises are defined as the bottom 3rd percentile of month-to-month log output declines. Data is generated via a 10,000 year simulation at the daily frequency, with the outcomes above then averaged to the monthly level. The solid blue line is the mean path, and the dotted blue lines represent the 25th and 75th percentiles. The thin horizontal line represents the unconditional average. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$. In this example, we set the sunspot drift $\mu_s = b_x x + 0.0002 s^{-1.5} - 0.0002(s_{\max} - s)^{-1.5}$, where $s_{\max} = 0.95$, $b_x = -1$, $\beta_x = 0.1$, and $\sigma_x = 0.025$. The parameters (β_x, σ_x) are approximately the values used for the mean-reversion and volatility of the diagnostic belief process in Maxted (2020).

First, jumps are large and sudden by definition, helping resolve the trouble with limited amplification. Second, the larger jumps that characterize a financial crisis can only happen from a moderate or good state that characterizes a boom. Third, introducing jumps reveals an additional indeterminacy that can be useful in exacerbating the previous point, namely the jump probability can be rigged so that it is more likely in good times.

Consider a broader class of solutions for the baseline model where capital price can also respond to an extrinsic jump shock, i.e.,

$$\frac{dq_t}{q_{t-}} = \mu_{q,t-} dt + \sigma_{q,t-} \cdot dZ_t - \ell_{q,t-} dJ_t,$$

where J is a Poisson process with intensity λ. For simplicity, we restrict attention to equilibria where the jump size ℓ_q is pre-determined, in particular a function of (η, q), and we focus on adverse jumps with $\ell_q \geq 0$.

We sketch the solution of a jumpy equilibrium (with more details in Online Appendix C.3). The risk-balance condition (RB) is modified to read

$$0 = \min\left[1-\kappa,\ \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}\left(|\sigma_R|^2 + \frac{\lambda \ell_q^2}{\left(1 - \frac{\kappa}{\eta}\ell_q\right)\left(1 - \frac{1-\kappa}{1-\eta}\ell_q\right)}\right)\right]. \tag{RBJ}$$

The additional terms involving ℓ_q arise because there is a jump risk premium. The price-output relation remains (PO).

By adding a new source of risk, we have an additional degree of freedom. The risk-balance condition disciplines overall risk—the term in parentheses of (RBJ) is pinned down given (η, q)—but the split between the Brownian and Poisson shocks is indeterminate. We have tremendous flexibility in our choice of ℓ_q.

It is easy to avoid stability concerns: just set $\ell_q = 0$ near the boundaries of the equilibrium region (i.e., the triangle in the (η, q) space in Figure 4). Doing this, the stability analysis remains unchanged from Theorem 1, since near the boundaries the economy behaves as if only hit by Brownian shocks.

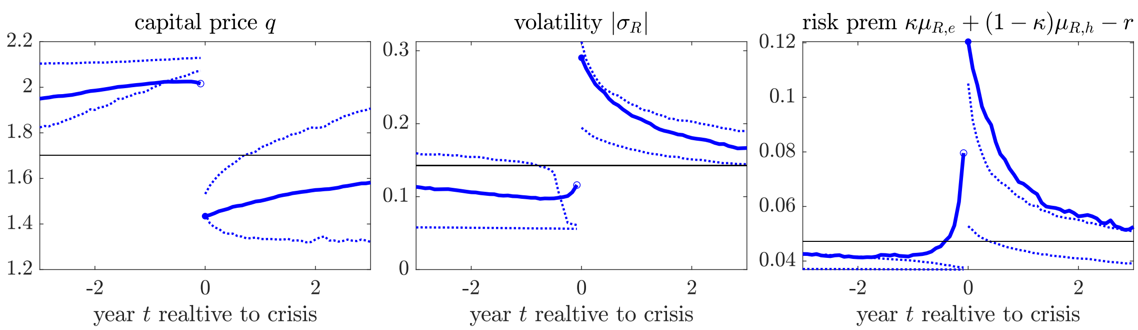

Figure 9: Event studies around financial crises in the jump-diffusion model. Crises are defined as the bottom 3rd percentile of month-to-month log output declines. Data is generated via a 100,000 year simulation at monthly frequency. The solid blue line is the mean, and dotted blue lines represent 25th and 75th percentiles. The horizontal black line is the unconditional mean. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$. We reflect (η, q) near boundaries of $\mathcal{D} := \{(\eta, q) : 0 < \eta < 1 \text{ and } \eta a_e + (1-\eta)a_h < q\bar{\rho}(\eta) \leq a_e\}$. Away from the boundaries, we set $\mu_q = 0.1(q^{\text{mid}}(\eta) - q)$, where q^{mid} corresponds to $\kappa(q^{\text{mid}}, \eta) = 0.8$.

Figure 9 shows a financial crisis event study from simulated data of the jump model. The solution and simulation method is described in detail in Online Appendix C.3. We make the following choice for jump sizes

$$\ell_q = \begin{cases} 0.95\ell_q^{\max}, & \text{if } \kappa > 0.9 \text{ and } 0.9\ell_q^{\max} > 0.2 \\ 0, & \text{otherwise,} \end{cases}$$

where $\ell_q^{\max}$ is the maximum allowable jump consistent with equilibrium (derived in the appendix). Thus, we focus attention on an economy with large jumps (greater than 20%) that are additionally only realized from efficient high-κ states.[27]

Because we focus on large jumps and only allow them in high-κ states, crises tend to arrive after a sequence of positive fundamental Brownian shocks. Accordingly, in the years before the crisis, asset prices are high, and both volatility and risk premia are

[27] In unreported results, we also solved an example without the $\kappa > 0.9$ restriction, i.e., where we set $\ell_q = 0.95\ell_q^{\max}\mathbf{1}_{\{0.9\ell_q^{\max} > 0.2\}}$. The results are similar to Figure 9—because large jumps still tend to happen from good states—but slightly muted.

below their usual level. Similar to Figure 8, volatility and risk premia tend to decline in the years prior to crisis. Crises arrive suddenly—with only a few months "warning" in terms of rising volatility and risk premia—and generate large movements in observables, because simulated crises often coincide with realizations of a jump.

5 Conclusion

We have shown that macroeconomic models with financial frictions may inherently permit sunspot volatility. The types of models we study are extremely common in macroeconomics, so this phenomenon cannot be ignored.

On the bright side, our paper demonstrates how a fully-rational notion of "sentiments" can be a powerful input into macro-finance dynamics. Unbounded amplification, sharp volatility spikes, and sentiment-driven boom-bust cycles are among the many interesting possibilities raised by our framework.

On the hazier side, our results suggest a modicum of caution. Many researchers employ numerical techniques to solve and analyze DSGE models that are built upon the core assumptions in our paper—these procedures implicitly select an equilibrium, without any explicit justification. A deeper analysis of refinements, perhaps leveraging global-games approaches or adaptive learning, still remains to be done.

What about policy?[28] Caveated by the need for further study on refinements, we can offer some initial thoughts. Some traditional policies become less effective in sunspot equilibria. For example, deposit insurance has less bite because run-like behavior can occur solely due to fire-sale coordination, i.e., on the asset side rather than the liability side. Sunspot equilibria also decouple financial crises from bank balance sheets and wealth, which defangs capital requirements, bailouts, and the like. Given the framework we study relies on fire sales, quantitative easing (e.g., asset purchases) could be interesting to analyze, perhaps even as an equilibrium refinement. Even speeches and commitments to future asset purchases might matter, by modifying beliefs about prices in extreme states and thus "calming the market."

[28] Many studies in the recent literature have moved toward policy analysis (Phelan, 2016; Dávila and Korinek, 2018; Drechsler et al., 2018; Di Tella, 2019; Silva, 2017; Elenev et al., 2021; Begenau, 2020; Begenau and Landvoigt, 2021; Klimenko et al., 2016).

References

Costas Azariadis. Self-fulfilling prophecies. *Journal of Economic Theory*, 25(3):380–396, 1981.

Costas Azariadis and Allan Drazen. Threshold externalities in economic development. *The Quarterly Journal of Economics*, 105(2):501–526, 1990.

Costas Azariadis and Bruce Smith. Financial intermediation and regime switching in business cycles. *The American Economic Review*, pages 516–536, 1998.

Costas Azariadis, Leo Kaas, and Yi Wen. Self-fulfilling credit cycles. *The Review of Economic Studies*, 83(4):1364–1405, 2016.

Philippe Bacchetta, Cédric Tille, and Eric Van Wincoop. Self-fulfilling risk panics. *The American Economic Review*, 102(7):3674–3700, 2012.

Nicholas Barberis, Robin Greenwood, Lawrence Jin, and Andrei Shleifer. X-capm: An extrapolative capital asset pricing model. *Journal of Financial Economics*, 115(1):1–24, 2015.

Matthew Baron and Wei Xiong. Credit expansion and neglected crash risk. *The Quarterly Journal of Economics*, 132(2):713–764, 2017.

Juliane Begenau. Capital requirements, risk choice, and liquidity provision in a business-cycle model. *Journal of Financial Economics*, 136(2):355–378, 2020.

Juliane Begenau and Tim Landvoigt. Financial regulation in a quantitative model of the modern banking system. Unpublished working paper, 2021.

Jess Benhabib and Pengfei Wang. Private information and sunspots in sequential asset markets. *Journal of Economic Theory*, 158:558–584, 2015.

Jess Benhabib, Pengfei Wang, and Yi Wen. Sentiments and aggregate demand fluctuations. *Econometrica*, 83(2):549–585, 2015.

Jess Benhabib, Xuewen Liu, and Pengfei Wang. Sentiments, financial markets, and macroeconomic fluctuations. *Journal of Financial Economics*, 120(2):420–443, 2016.

Jess Benhabib, Xuewen Liu, and Pengfei Wang. Financial markets, the real economy, and self-fulfilling uncertainties. *The Journal of Finance*, 74(3):1503–1557, 2019.

Jess Benhabib, Xuewen Liu, and Pengfei Wang. Self-fulfilling risk panics: An expected utility framework. Unpublished working paper, 2020.

Markus K Brunnermeier and Yuliy Sannikov. A macroeconomic model with a financial sector. *The American Economic Review*, 104(2):379–421, 2014.

Markus K Brunnermeier and Yuliy Sannikov. Macro, money, and finance: A continuous-time approach. *Handbook of Macroeconomics*, 2:1497–1545, 2016.

Ricardo J Caballero and Alp Simsek. A risk-centric model of demand recessions and speculation. *The Quarterly Journal of Economics*, 135(3):1493–1566, 2020.

David Cass and Karl Shell. Do sunspots matter? *Journal of Political Economy*, 91(2): 193–227, 1983.

Eduardo Dávila and Anton Korinek. Pecuniary externalities in economies with financial frictions. *The Review of Economic Studies*, 85(1):352–395, 2018.

Sebastian Di Tella. Uncertainty shocks and balance sheet recessions. *Journal of Political Economy*, 125(6):2038–2081, 2017.

Sebastian Di Tella. Optimal regulation of financial intermediaries. *The American Economic Review*, 109(1):271–313, 2019.

Itamar Drechsler, Alexi Savov, and Philipp Schnabl. A model of monetary policy and risk premia. *The Journal of Finance*, 73(1):317–373, 2018.

Vadim Elenev, Tim Landvoigt, and Stijn Van Nieuwerburgh. A macroeconomic model with financially constrained producers and intermediaries. *Econometrica*, 89(3):1361–1418, 2021.

Emmanuel Farhi and Jean Tirole. Bubbly liquidity. *The Review of Economic Studies*, 79(2): 678–706, 2012.

Roger EA Farmer and Jess Benhabib. Indeterminacy and increasing returns. *Journal of Economic Theory*, 63:19–41, 1994.

David M Frankel, Stephen Morris, and Ady Pauzner. Equilibrium selection in global games with strategic complementarities. *Journal of Economic Theory*, 108(1):1–44, 2003.

Nicolae Gârleanu and Stavros Panageas. What to expect when everyone is expecting: Self-fulfilling expectations and asset-pricing puzzles. *Journal of Financial Economics*, 140(1):54–73, 2021.

Mark Gertler and Nobuhiro Kiyotaki. Financial intermediation and credit policy in business cycle analysis. *Handbook of Monetary Economics*, 3(3):547–599, 2010.

Mark Gertler and Nobuhiro Kiyotaki. Banking, liquidity, and bank runs in an infinite horizon economy. *The American Economic Review*, 105(7):2011–43, 2015.

Mark Gertler, Nobuhiro Kiyotaki, and Andrea Prestipino. A macroeconomic model with financial panics. *The Review of Economic Studies*, 87(1):240–288, 2020.

Chao Gu, Fabrizio Mattesini, Cyril Monnet, and Randall Wright. Endogenous credit cycles. *Journal of Political Economy*, 121(5):940–965, 2013.

John C Harsanyi and Reinhard Selten. A general theory of equilibrium selection in games. *MIT Press Books*, 1, 1988.

Zhiguo He and Arvind Krishnamurthy. A model of capital and crises. *The Review of Economic Studies*, 79(2):735–777, 2012.

Zhiguo He and Arvind Krishnamurthy. Intermediary asset pricing. *The American Economic Review*, 103(2):732–70, 2013.

Zhiguo He and Arvind Krishnamurthy. A macroeconomic framework for quantifying systemic risk. *American Economic Journal: Macroeconomics*, 11(4):1–37, 2019.

Julien Hugonnier. Rational asset pricing bubbles and portfolio constraints. *Journal of Economic Theory*, 147(6):2260–2302, 2012.

Òscar Jordà, Moritz Schularick, and Alan M Taylor. Financial crises, credit booms, and external imbalances: 140 years of lessons. *IMF Economic Review*, 59(2):340–378, 2011.

Òscar Jordà, Moritz Schularick, and Alan M Taylor. When credit bites back. *Journal of Money, Credit and Banking*, 45(s2):3–28, 2013.

Òscar Jordà, Moritz Schularick, and Alan M Taylor. Betting the house. *Journal of International Economics*, 96:S2–S18, 2015a.

Òscar Jordà, Moritz Schularick, and Alan M Taylor. Leveraged bubbles. *Journal of Monetary Economics*, 76:S1–S20, 2015b.

Òscar Jordà, Björn Richter, Moritz Schularick, and Alan M Taylor. Bank capital redux: solvency, liquidity, and crisis. *The Review of Economic Studies*, 88(1):260–286, 2021.

Rafail Khasminskii. *Stochastic stability of differential equations*, volume 66. Springer Science & Business Media, 2011.

Paymon Khorrami. Entry and slow-moving capital: using asset markets to infer the costs of risk concentration. Unpublished working paper. Imperial College London, 2018.

Paymon Khorrami and Fernando Mendo. Fear and volatility at the zero lower bound. Unpublished working paper. Imperial College London, 2021.

Paymon Khorrami and Alexander Zentefis. Arbitrage and beliefs. Unpublished working paper, CESifo, 2020.

Nobuhiro Kiyotaki and John Moore. Credit cycles. *Journal of Political Economy*, 105(2): 211–248, 1997.

Nataliya Klimenko, Sebastian Pfeil, Jean-Charles Rochet, and Gianni De Nicolò. Aggregate bank capital and credit dynamics. Unpublished working paper. University of Zurich, 2016.

Nataliya Klimenko, Sebastian Pfeil, and Jean-Charles Rochet. A simple macroeconomic model with extreme financial frictions. *Journal of Mathematical Economics*, 68:92–102, 2017.

Narayana R Kocherlakota. Bubbles and constraints on debt accumulation. *Journal of Economic Theory*, 57(1):245–256, 1992.

Arvind Krishnamurthy and Wenhao Li. Dissecting mechanisms of financial crises: Intermediation and sentiment. Unpublished working paper, 2020.

Arvind Krishnamurthy and Tyler Muir. How credit cycles across a financial crisis. Unpublished NBER working paper, 2017.

Nicolai V Krylov. Diffusion on a plane with reflection. construction of the process. *Siberian Mathematical Journal*, 10(2):244–252, 1969.

Nicolai V Krylov. On weak uniqueness for some diffusions with discontinuous coefficients. *Stochastic Processes and their Applications*, 113(1):37–64, 2004.

Zheng Liu and Pengfei Wang. Credit constraints and self-fulfilling business cycles. *American Economic Journal: Macroeconomics*, 6(1):32–69, 2014.

David López-Salido, Jeremy C Stein, and Egon Zakrajšek. Credit-market sentiment and the business cycle. *The Quarterly Journal of Economics*, 132(3):1373–1426, 2017.

Rodolfo Manuelli and James Peck. Sunspot-like effects of random endowments. *Journal of Economic Dynamics and Control*, 16(2):193–206, 1992.

Peter Maxted. A macro-finance model with sentiment. Unpublished working paper. Harvard University, 2020.

Fernando Mendo. Risky low-volatility environments and the stability paradox. Unpublished working paper, 2020.

Atif Mian, Amir Sufi, and Emil Verner. Household debt and business cycles worldwide. *The Quarterly Journal of Economics*, 132(4):1755–1817, 2017.

Jianjun Miao and Pengfei Wang. Asset bubbles and credit constraints. *The American Economic Review*, 108(9):2590–2628, 2018.

Alan Moreira and Alexi Savov. The macroeconomics of shadow banking. *The Journal of Finance*, 72(6):2381–2432, 2017.

Tyler Muir. Financial crises and risk premia. *The Quarterly Journal of Economics*, 132(2): 765–809, 2017.

Gregory Phelan. Financial intermediation, leverage, and macroeconomic instability. *American Economic Journal: Macroeconomics*, 8(4):199–224, 2016.

Thomas Piketty. The dynamics of the wealth distribution and the interest rate with credit rationing. *The Review of Economic Studies*, 64(2):173–189, 1997.

Carmen M Reinhart and Kenneth S Rogoff. *This time is different: Eight centuries of financial folly*. Princeton University Press, 2009.

Carmen M Reinhart and Kenneth S Rogoff. Recovery from financial crises: Evidence from 100 episodes. *The American Economic Review*, 104(5):50–55, 2014.

Jose A Scheinkman and Laurence Weiss. Borrowing constraints and aggregate economic activity. *Econometrica*, 54(1):23–45, 1986.

Dejanir Silva. The risk channel of unconventional monetary policy. Unpublished working paper. University of Illinois., 2017.

Online Appendix:
Rational Sentiments and Financial Frictions

Paymon Khorrami and Fernando Mendo

October 6, 2021

A Proofs for Section 2

Proof of Lemma 1. Suppose $\kappa = 1$, $q = a_e/\bar{\rho}$, and $\sigma_q = 0$. Set μ_η and σ_η by (11)-(12), and set r by (9). By inspection, both (PO) and (RB) are satisfied. Furthermore, the Itô condition $\sigma_q = \frac{q'}{q}\sigma_\eta$ is trivially satisfied. Thus, Definition 1 is satisfied. □

Proof of Proposition 1. Consequence of Proposition D.1 (take $\kappa_0 \to 0$). □

Proof of Lemma 2. In the text leading up to the statement of the lemma. □

Proof of Lemma 3. Note that the other equations characterizing equilibrium, beyond (18), are (PO) and (RB), the latter repeated here for convenience:

$$0 = \min\Big[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}(\sigma + \sigma_q)^2\Big]. \tag{A.1}$$

Denote the equilibrium solution for $\sigma > 0$ by $(q^{(\sigma)}, \kappa^{(\sigma)})$. Define $q^{(0)} := \lim_{\sigma\to 0} q^{(\sigma)}$ and $\kappa^{(0)} := \lim_{\sigma \to 0}\kappa^{(\sigma)}$. Combine equations (18) and (A.1) and rearrange terms to get

$$\Big(1 - (\kappa^{(\sigma)} - \eta)\frac{(q^{(\sigma)})'}{q^{(\sigma)}}\Big)^2 = \frac{(\kappa^{(\sigma)} - \eta) q^{(\sigma)}}{\eta(1-\eta)(a_e - a_h)}\sigma, \quad \text{if} \quad \kappa^{(\sigma)} < 1. \tag{A.2}$$

Note that this implies $\kappa^{(\sigma)} > \eta$. Furthermore, continuity of $\kappa^{(\sigma)}(\eta)$ and $\kappa_0 = \kappa^{(\sigma)}(0+) < 1$ imply $\kappa^{(\sigma)}(\eta) < 1$ for all η close enough to 0. Using these facts, and writing (A.2) instead as an integral equation, we obtain

$$\frac{q^{(\sigma)}(\eta_2)}{q^{(\sigma)}(\eta_1)} = \exp\Big\{\int_{\eta_1}^{\eta_2} \frac{1}{\kappa^{(\sigma)}(x) - x}\Big[1 \pm \sqrt{\frac{(\kappa^{(\sigma)}(x) - x) q^{(\sigma)}(x)}{x(1-x)(a_e - a_h)}\sigma}\Big] dx\Big\}, \quad 0 < \eta_1 < \eta_2,$$

where η_2 is chosen small enough. Because the right-hand-side is continuous in both $q^{(\sigma)}$ and $\kappa^{(\sigma)}$, and both are bounded, taking the limit as $\sigma \to 0$ implies

$$\frac{q^{(0)}(\eta_2)}{q^{(0)}(\eta_1)} = \exp\left\{ \int_{\eta_1}^{\eta_2} \frac{1}{\kappa^{(0)}(x) - x} dx \right\}.$$

Differentiate this equation with respect to η_2 to obtain

$$\frac{d}{d\eta} \log q^{(0)} = \frac{1}{\kappa^{(0)} - \eta},$$

for all η small enough. Rearranging this equation delivers the ODE characterizing the W-BSE, i.e., selecting the solution $(\kappa - \eta)q'/q = 1$ in equation (14). Since $\kappa^{(\sigma)}(0+) = \kappa_0$ is fixed for all $\sigma > 0$, we also have the desired boundary condition $\kappa^{(0)}(0+) = \kappa_0$, for any $\kappa_0 \in [0, 1)$. Finally, all the other equations of the W-BSE can be verified by simply taking limits as $\sigma \to 0$. □

B Proofs for Section 3

B.1 Proof of Theorem 1

Step 0: Reduce the system. We will start by eliminating $(r, \kappa, \sigma_\eta, \mu_\eta)$ from the system of endogenous objects, given $(\eta, q, \sigma_q, \mu_q)$. First, price-output relation (PO) determines κ as a function of (η, q) and nothing else, given by

$$\kappa(\eta, q) := \frac{q\bar{\rho}(\eta) - a_h}{a_e - a_h}. \tag{B.1}$$

Second, substituting this result for κ, equation (9) fully determines r, given knowledge of $(\eta, q, \sigma_q, \mu_q)$. Third, equations (11)-(12), after plugging in the result for κ, fully determine (σ_η, μ_η), given knowledge of (η, q, σ_q). Thus, given (η, q), it suffices to determine (σ_q, μ_q) from the remaining conditions, namely (RB).

Step 1: Define perturbed domain. To facilitate analysis, it will be convenient to analyze a slightly modified system instead of (η, q), and on a perturbed domain.

First, define the following auxiliary functions. Fix $\epsilon \in (0, \frac{a_e - a_h}{\rho_h})$. Let $\beta(\cdot)$ be a strictly increasing, continuously differentiable function such that $\beta(1) = -\beta(0) = \epsilon$, and $\beta(\eta^*_\beta) = 0$, where $\eta^*_\beta \in (\eta^*, 1)$ and

$$\eta^* := \frac{\rho_h}{\rho_e}\Big(\frac{1 - a_h/a_e}{\sigma^2}\rho_e - 1 + \frac{\rho_h}{\rho_e}\Big)^{-1}. \tag{B.2}$$

Note that $\eta^* < 1$ by Assumption 1, part (ii). Let $\alpha(\cdot)$ be an increasing, continuously differentiable function such that $\alpha(0) = 0$, $\alpha'(0) \in (0, \infty)$, and $\alpha(1) = \epsilon/2$.

Next, define the following functions,

$$\begin{aligned} q^H(\eta) &:= a_e/\bar{\rho}(\eta) \\ q^L(\eta) &:= \bar{a}(\eta)/\bar{\rho}(\eta), \end{aligned}$$

where $\bar{a}(\eta) := \eta\rho_e + (1-\eta)\rho_h$. Using (B.1), one notices that q^H corresponds to the capital price when $\kappa = 1$, whereas q^L corresponds to the capital price when $\kappa = \eta$. Put

$$\begin{aligned} q^H_\beta(\eta) &:= q^H(\eta) + \beta(\eta) \\ q^L_\alpha(\eta) &:= q^L(\eta) + \alpha(\eta). \end{aligned}$$

Using these functions, define the perturbed domain (which is an open set)

$$\mathcal{X} := \Big\{(\eta, x) \,:\, \eta \in (0,1) \quad \text{and} \quad q^L_\alpha(\eta) < x < q^H_\beta(\eta)\Big\}.$$

Note that, boundaries aside, $\mathcal{X}$ will coincide with $\mathcal{D}$ as $\epsilon \to 0$. For reference, the perturbed domain $\mathcal{X}$ is displayed in Figure B.1.

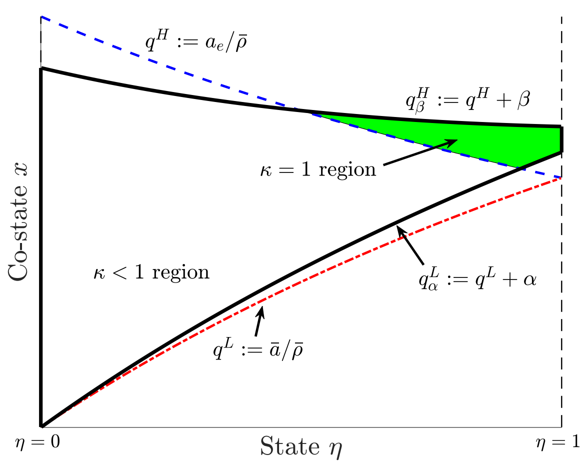

Figure B.1: The perturbed domain $\mathcal{X}$ is shown as the region surrounded by solid black lines. The original domain $\mathcal{D}$ is the region defined by the dashed lines. The perturbation functions α and β are chosen to be linear functions, with $\epsilon = 0.2$. Parameters: $\rho_e = 0.07$, $\rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$.

We will define a stochastic process x_t such that the capital price q coincides with x

when it lies below q^H, i.e.,

$$q_t = \min\Big[x_t,\, q^H(\eta_t)\Big]. \tag{B.3}$$

By (B.3), we may analyze the dynamical system $(\eta_t, x_t)_{t\geq 0}$ rather than $(\eta_t, q_t)_{t\geq 0}$. Furthermore, to prove the claim that $(\eta_t, q_t)_{t\geq 0}$ remains in $\mathcal{D}$ almost-surely, it suffices to prove $(\eta_t, x_t)_{t\geq 0}$ remains in $\mathcal{X}$ almost-surely (Step 4 below).

Step 2: Construct σ_q so that (RB) is satisfied. First consider $\{x < q^H(\eta)\}$ so that $q = x$. Note that this case corresponds to $\kappa < 1$. Let $\gamma(\eta, x) : \mathcal{X} \mapsto (0,1)$ be any C^1 function. Put

$$\sigma_q = \begin{bmatrix} \sqrt{\gamma \frac{\eta(1-\eta)}{\kappa-\eta}\frac{a_e-a_h}{q}} - \sigma \\ \sqrt{(1-\gamma)\frac{\eta(1-\eta)}{\kappa-\eta}\frac{a_e-a_h}{q}} \end{bmatrix}, \quad \text{if } x < q^H(\eta). \tag{B.4}$$

Substituting (B.4), one can verify that the second term of condition (RB) is zero. Importantly, the definitions of q^L_α and q^H_β imply that σ_q is bounded on $\mathcal{X} \cap \{x < q^H(\eta)\}$. Indeed, because of $\alpha'(0) > 0$, the slowest possible rate that $\kappa \to 0$ as $\eta \to 0$ is lower-bounded away from 1, i.e., $\liminf_{\eta\to 0, (\eta,x)\in\mathcal{X}} \kappa/\eta > 1$. And because $\alpha(1) > 0$, we have $\kappa = 1$ for all η near enough to 1; thus η is bounded away from 1 on $\{x < q^H(\eta)\}$.

Next consider $\{x \geq q^H(\eta)\}$ so that $q = q^H(\eta)$. Note that this case corresponds to $\kappa = 1$. Since q is an explicit function of η, we use Itô's formula to compute $\binom{1}{0} \cdot \sigma_q = -\sigma_\eta \bar{\rho}'/\bar{\rho}$, which after substituting equation (12) for σ_η delivers

$$\sigma_q = \begin{bmatrix} -\frac{(1-\eta)(\rho_e-\rho_h)/\bar{\rho}}{1+(1-\eta)(\rho_e-\rho_h)/\bar{\rho}}\sigma \\ 0 \end{bmatrix}, \quad \text{if } x \geq q^H(\eta). \tag{B.5}$$

Note that (B.5) will be consistent with (RB) as long as $(\eta_t, x_t)_{t\geq 0}$ remains in $\mathcal{X}$ almost-surely, which will be verified in Step 4.[29]

Note finally that σ_q defined in (B.4)-(B.5) is solely a function of (η, x), so sometimes we will write $\sigma_q(\eta, x)$. Similarly, with σ_q in hand, we now have μ_η and σ_η as functions of (η, x) alone.

Step 3: Construct μ_q. Similar to σ_q, separately consider $\{x < q^H(\eta)\}$ and $\{x \geq q^H(\eta)\}$.

[29] Plugging $q = a_e/\bar{\rho}$ into the second term of equation (RB), we require $|\sigma_R|^2 \leq \eta\bar{\rho}(\eta)(1 - a_h/a_e)$. Substituting (B.5), we obtain $|\sigma_R|^2 = \sigma^2(\bar{\rho}/\rho_e)^2$. Combining these, we require. $\eta \geq \eta^*$ when $x \geq q^H(\eta)$, where η^* is defined in (B.2). Therefore, for all $\eta < \eta^*$, we insist $x < q^H(\eta)$. As long as $(\eta, x) \in \mathcal{X}$, this will hold, because $q^H_\beta(\eta) < q^H(\eta)$ for all $\eta < \eta^*$, and $x < q^H_\beta(\eta)$ for all η.

On $\{x \geq q^H(\eta)\}$, since $q = q^H(\eta)$ is an explicit function of η, we set μ_q via Itô's formula. On $\{x < q^H(\eta)\}$, we have no equilibrium considerations restricting μ_q. Thus, we will put $\mu_q = m_q$, where m_q is a function in class $\mathcal{M}$, defined as follows. A function $m : \mathcal{X} \mapsto \mathbb{R}$ is a member of $\mathcal{M}$ if m is C^1 and possesses the following boundary conditions:

$$\inf_{\eta \in (0,1)} \lim_{x \searrow q_\alpha^L(\eta)} \Big(x - q_\alpha^L(\eta)\Big) m(\eta, x) = +\infty \tag{B.6}$$

$$\sup_{\eta \in (0,1)} \lim_{x \nearrow q_\beta^H(\eta)} \Big(q_\beta^H(\eta) - x\Big) m(\eta, x) = -\infty \tag{B.7}$$

$$\text{for any } x \in (q_\alpha^L(0), q_\beta^H(0)), \quad \lim_{\eta \searrow 0} |m(\eta, x)| < +\infty \tag{B.8}$$

$$\text{for any } x \in (q_\alpha^L(1), q_\beta^H(1)), \quad \lim_{\eta \nearrow 1} |m(\eta, x)| < +\infty. \tag{B.9}$$

Collecting these results

$$\mu_q(\eta, x) = \begin{cases} m_q(\eta, x), & \text{if } x < q^H(\eta); \\ \frac{\rho_e - \rho_h}{\bar{\rho}(\eta)^2}[-\bar{\rho}(\eta)\mu_\eta(\eta, x) + |\sigma_\eta(\eta, x)|^2], & \text{if } x \geq q^H(\eta). \end{cases} \tag{B.10}$$

Step 4: Verify stationarity. We demonstrate the time-paths $(\eta_t, x_t)_{t \geq 0}$ remain in $\mathcal{X}$ almost-surely and admit a stationary distribution.

The dynamics of x_t are specified as follows. Denote its diffusion and drift coefficients by $(x\sigma_x, x\mu_x)$, where σ_x and μ_x are functions of (η, x) to be specified shortly. By (B.3), when $q_\alpha^L(\eta) < x < q^H(\eta)$, we must put $\sigma_x = \sigma_q$ and $\mu_x = \mu_q$. Outside of this region, we put σ_x and μ_x to preserve stationarity.

To this end, let $\tilde{\sigma}_x : \mathcal{X} \mapsto \mathbb{R}_+$ be any positive, bounded, C^1 function.[30] Put

$$\sigma_x(\eta, x) = \begin{cases} \sigma_q(\eta, x), & \text{if } x < q^H(\eta); \\ \tilde{\sigma}_x(\eta, x), & \text{if } x \geq q^H(\eta). \end{cases}$$

Note that σ_x is bounded (recall σ_q is bounded, and $\tilde{\sigma}_x$ is assumed bounded).

Similarly, for the drift, let $m_x : \mathcal{X} \mapsto \mathbb{R}$ be any function in class $\mathcal{M}$ defined above

[30]Note that $\tilde{\sigma}_x$ need not vanish at the boundary of $\mathcal{X}$, but if it does some of the boundary conditions on m_x, to follow, can be relaxed.

(note: m_x need not coincide with m_q above). Put

$$\mu_x(\eta, x) = \begin{cases} \mu_q(\eta, x), & \text{if } x < q^H(\eta); \\ m_x(\eta, x), & \text{if } x \geq q^H(\eta). \end{cases}$$

Thus, μ_x satisfies boundary conditions (B.6)-(B.9) on all boundaries of $\mathcal{X}$.

Corresponding to the SDEs induced by $(\sigma_\eta, \sigma_x, \mu_\eta, \mu_x)$, define the infinitesimal generator $\mathscr{L}$, where for any C^2 function f,

$$\mathscr{L} f = \mu_\eta \partial_\eta f + (x\mu_x)\partial_x f + \frac{1}{2}|\sigma_\eta|^2 \partial_{\eta\eta} f + \frac{1}{2}|x\sigma_x|^2 \partial_{xx} f + x\sigma_x \cdot \sigma_\eta \partial_{\eta x} f.$$

Let $\{\mathcal{X}_n\}_{n\geq 1}$ be an increasing sequence of open sets, whose closures are contained in $\mathcal{X}$, such that $\cup_{n\geq 1}\mathcal{X}_n = \mathcal{X}$. Note that $(\sigma_\eta, \sigma_x, \mu_\eta, \mu_x)$ are bounded on $\mathcal{X}_n$ for each n. Consequently, despite the (potential) discontinuity in $(\sigma_\eta, \sigma_x, \mu_\eta, \mu_x)$ at the one-dimensional subset $\{x = q^H(\eta)\}$, there exists a unique weak solution $(\tilde{\eta}_t^n, \tilde{x}_t^n)_{0\leq t\leq \tau_n}$, up to first exit time $\tau_n := \inf\{t : (\eta_t, x_t) \notin \mathcal{X}_n\}$, to the SDEs defined by the infinitesimal generator $\mathscr{L}$. See Krylov (1969, 2004) for weak existence and uniqueness in the presence of drift and diffusion discontinuities.

Letting $\tau := \lim_{n\to\infty} \tau_n$, we thus define $(\eta_t, x_t)_{0\leq t\leq \tau}$ by piecing $(\tilde{\eta}_t^n, \tilde{x}_t^n)_{0\leq t\leq \tau_n}$ together for successive n. In other words, $(\eta_t, x_t) = (\tilde{\eta}_t^n, \tilde{x}_t^n)$ for $0 \leq t \leq \tau_n$, each n. Our goal is to show (a) $\tau = +\infty$ a.s.; and (b) the resulting stochastic process $(\eta_t, x_t)_{t\geq 0}$ possesses a non-degenerate stationary distribution on $\mathcal{X}$. These will be proved if we can obtain a function v satisfying Lemma B.1 below.

Define the positive function v by

$$v(\eta, x) := \frac{1}{\eta^{1/2}} + \frac{1}{1-\eta} + \frac{1}{x - q_\alpha^L(\eta)} + \frac{1}{q_{\beta,\lambda}^H(\eta) - x}.$$

Note that v diverges to $+\infty$ at the boundaries of $\mathcal{X}$, so assumption (i) of Lemma B.1 is proved. Next, if assumption (iii) of Lemma B.1 holds (which we will prove below), then there exists N such that $\mathscr{L}v < 0$ on $\mathcal{X}\backslash\mathcal{X}_n$ for all $n > N$. Furthermore, for each given n, $\mathscr{L}v$ is bounded on $\mathcal{X}_n$. Consequently, we can find a constant c large enough such that $\mathscr{L}v \leq cv$ on all of $\mathcal{X}$, which verifies part (ii) of Lemma B.1.

It remains to prove assumption (iii) of Lemma B.1, namely that $\mathscr{L}v \to -\infty$ as $(\eta, x) \to \partial\mathcal{X}$. We will examine the boundaries of $\mathcal{X}$ one-by-one. In the following, we use the notation $g(x) = o(f(x))$ if $g(x)/f(x) \to 0$ as $x \to 0$, and the notation $g(x) = O(f(x))$ if $g(x)/f(x) \to C$ as $x \to 0$, where C is a finite constant.

As $\eta \to 0$ (and x bounded away from $q^L_\alpha(0)$ and $q^H_\beta(0)$, such that κ is bounded away from 0 and 1, the latter due to the definition of q^H_β), we have

$$\mu_\eta = \delta\nu + \frac{a_e - a_h}{x}\kappa + \eta[\rho_h - \rho_e - \delta] + o(\eta) \quad \text{and} \quad |\sigma_\eta|^2 = \eta(\kappa - \eta)\frac{a_e - a_h}{x} + o(\eta)$$
$$\mu_x = O(1) \quad \text{and} \quad |\sigma_x|^2 = O(1).$$

We used condition (B.8) to obtain μ_x bounded. Thus,

$$\mathscr{L}v = -\frac{1}{2\eta^{3/2}}[\delta\nu + \frac{1}{4}\frac{a_e - a_h}{x}\kappa] + o(\eta^{-3/2}) \to -\infty,$$

irrespective of $\delta\nu > 0$ or $\delta\nu = 0$.

As $\eta \to 1$ (and x bounded away from $q^L_\alpha(1)$ and $q^H_\beta(1)$; note that $\kappa = 1$ at this boundary), we have

$$\mu_\eta = -\delta(1-\nu) - (\rho_e - \rho_h)(1-\eta) + o(1-\eta) \quad \text{and} \quad |\sigma_\eta|^2 = (1-\eta)^2\sigma^2$$
$$\mu_x = O(1) \quad \text{and} \quad |\sigma_x|^2 = O(1).$$

We used condition (B.9) to obtain μ_x bounded. Thus,

$$\mathscr{L}v = -(1-\eta)^{-2}\delta(1-\nu) - (1-\eta)^{-1}[\rho_e - \rho_h - \sigma^2] + o((1-\eta)^{-1}) \to -\infty,$$

irrespective of $\delta(1-\nu)$, due to Assumption 1 part (iii).

We separately calculate the limit $x \to q^L_\alpha(\eta)$ (with η bounded away from 0) in the two cases $\{x < q^H(\eta)\}$ and $\{x \geq q^H(\eta)\}$, since $\kappa < 1$ in the first case, and $\kappa = 1$ in the second case. Still, we find that in both cases,

$$\mu_\eta = O(1) \quad \text{and} \quad |\sigma_\eta|^2 = O(1)$$
$$\mu_x = o((x - q^L_\alpha)^{-1}) \quad \text{and} \quad |\sigma_x|^2 = O(1).$$

We used condition (B.6) to obtain the order of μ_x. Thus,

$$\mathscr{L}v = -(x - q^L_\alpha)^{-2}x\mu_x + O((x - q^L_\alpha)^{-3}) \to -\infty.$$

Similarly, we separately calculate the limit $x \to q^H_\beta(\eta)$ (with η bounded away from 0) in the two cases $\{x < q^H(\eta)\}$ and $\{x \geq q^H(\eta)\}$. Again, we find that in both cases,

$$\mu_\eta = O(1) \quad \text{and} \quad |\sigma_\eta|^2 = O(1)$$
$$\mu_x = (-1) \times o((q^H_\beta - x)^{-1}) \quad \text{and} \quad |\sigma_x|^2 = O(1).$$

We used condition (B.7) to obtain the order of μ_x. Thus,

$$\mathscr{L}v = (q^H_\beta - x)^{-2} x\mu_x + O((q^H_\beta - x)^{-3}) \to -\infty.$$

Finally, all the corners of $\mathcal{X}$ can be analyzed in a straightforward way by combining the cases above, with the exception of $(\eta, x) = (0, q^L_\alpha(0)) = (0, a_h/\rho_h)$. Approaching this corner, we must take a particular path of $x \to a_h/\rho_h$ as $\eta \to 0$. Denote this path by $\hat{x}(\eta)$ and denote the asymptotic slope by $\hat{x}'(0) \in (\frac{d}{d\eta} q^L_\alpha(0), +\infty)$, where $\frac{d}{d\eta} q^L_\alpha(0) = [\frac{a_e}{a_h} - \frac{\rho_e}{\rho_h}]\frac{a_h}{\rho_h} + \alpha'(0) > 0$, by Assumption 1, part (i), and the fact that $\alpha'(0) > 0$. Denote the associated path of κ by $\hat{\kappa}(\eta)$ and the corresponding asymptotic slope by $\hat{\kappa}'(0) = \frac{1}{a_e - a_h}[\hat{x}'(0)\rho_h + (\rho_e - \rho_h)a_h/\rho_h]$. Substituting in, we find $\hat{\kappa}'(0) \in (1 + \frac{\alpha'(0)}{a_e - a_h}, +\infty)$. When computing $\mathscr{L}v$, we will take the supremum over all possible paths, meaning over $\hat{x}'(0)$ and $\hat{\kappa}'(0)$. Using similar calculations from the initial $\eta \to 0$ case, but using these paths, we obtain

$$\mu_\eta = \delta v + \eta[\frac{a_e - a_h}{\hat{x}}\hat{\kappa}' + \rho_h - \rho_e - \delta] + o(\eta) \quad \text{and} \quad |\sigma_\eta|^2 = \eta^2[\hat{\kappa}' - 1]\frac{a_e - a_h}{\hat{x}} + o(\eta)$$
$$\mu_x = o((\hat{x} - q^L_\alpha)^{-1}) \quad \text{and} \quad |\sigma_x|^2 = O(1)$$
$$\text{and} \quad \sigma_x \cdot \sigma_\eta = \eta[\frac{a_e - a_h}{\hat{x}} - \sigma(\gamma(\hat{\kappa}' - 1)\frac{a_e - a_h}{\hat{x}})^{1/2}] + o(\eta).$$

Since $\hat{x} \geq O(\eta)$ and $\hat{\kappa} \geq O(\eta)$ (in the sense that both could be $+\infty$), we may treat terms like $(\hat{x} - q^L_\alpha)^{-1}$ as smaller than η^{-1}. This identifies the dominant terms as those associated to μ_η, $|\sigma_\eta|^2$, and μ_x. Thus,

$$\begin{aligned}\mathscr{L}v = &-\frac{1}{2\eta^{3/2}}\delta v + \frac{1}{2\eta^{1/2}}[\rho_e - \rho_h + \delta - \frac{a_e - a_h}{\hat{x}} - \frac{a_e - a_h}{\hat{x}}(\hat{\kappa}' - 1)/4] + o(\eta^{-3/2}) \\ &- (\hat{x} - q^L_\alpha)^{-2} x\mu_x + O((\hat{x} - q^L_\alpha)^{-3}) \to -\infty,\end{aligned}$$

irrespective of δv, because $\rho_e - \rho_h - \frac{a_e - a_h}{a_h/\rho_h} = \rho_h[\rho_e/\rho_h - a_e/a_h] < 0$ by Assumption 1, part (i), and because $\inf\{\hat{\kappa}'(0)\} > 1$.

This completes the verification that $\mathscr{L}v \to -\infty$ as $(\eta, x) \to \partial\mathcal{X}$, which proves stationarity by Lemma B.1 below. This completes the proof. $\square$

B.2 Stochastic stability: a useful lemma

To prove the stationarity claims of Theorem 1 and Proposition 2, we need the following lemma, which is a slight generalization of Theorems 3.5 and 3.7 of Khasminskii (2011), in the sense that weaker conditions are imposed on the coefficients α and β. Indeed, any coefficients (α, β) are permissible as long as they admit existence of a weak solution to the SDE system. See also Remark 3.5 and Corollary 3.1 in Khasminskii (2011) which allow the arguments in $\mathbb{R}^l$ to generalize to any open domain $\mathcal{D}$.

Lemma B.1. *Suppose $(X_t)_{0\le t\le\tau}$ is a weak solution to the SDE $dX_t = \beta(X_t)dt + \alpha(X_t)dZ_t$ in an open connected domain $\mathcal{D} \subset \mathbb{R}^l$, where Z is a d-dimensional Brownian motion and $\tau := \inf\{t : X_t \notin \mathcal{D}\}$ is the first exit time from $\mathcal{D}$. Define the infinitesimal generator $\mathscr{L}$ by (for any C^2 function f)*

$$\mathscr{L} f = \sum_{i=1}^{n} \beta_i \frac{\partial f}{\partial x_i} f + \frac{1}{2}\sum_{i,j=1}^{n} (\alpha_i \cdot \alpha_j) \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Suppose there is a non-negative C^2 function $v : \mathcal{D} \mapsto \mathbb{R}_+$ such that (i) $\liminf_{x\to\partial\mathcal{D}} v(x) = +\infty$; (ii) $\mathscr{L}v \le cv$ for some constant $c \ge 0$; and (iii) $\limsup_{x\to\partial\mathcal{D}} \mathscr{L}v(x) = -\infty$. Then,

- *(a) $\tau = +\infty$ almost-surely;*
- *(b) the distribution of X_0 can be chosen such that $(X_t)_{t\ge0}$ is stationary.*

Proof of Lemma B.1. Let $\{\mathcal{D}_n\}_{n\ge1}$ be an increasing sequence of open sets, whose closures are contained in $\mathcal{D}$, such that $\cup_{n\ge1}\mathcal{D}_n = \mathcal{D}$. Let $\tau_n := \inf\{t : X_t \notin \mathcal{D}_n\}$, and note that $\tau = \lim_{n\to\infty} \tau_n$ is the monotone limit of these exit times. Define $w(t, x) := v(x)\exp(-ct)$, which satisfies $\mathscr{L}w \le 0$ by assumption (ii). Using Itô's formula, we have

$$\mathbb{E}[v(X_{\tau_n\wedge t})e^{-c(\tau_n\wedge t)} - v(X_0)] = \mathbb{E}\int_0^{\tau_n\wedge t} \mathscr{L}w(u, X_u)du \le 0.$$

Since $(\tau_n \wedge t) \le t$ and $v \ge 0$, we obtain

$$\mathbb{E}[v(X_{\tau_n\wedge t})] \le e^{ct}\mathbb{E}[v(X_0)].$$

Because $\mathbb{E}[v(X_{\tau_n\wedge t})] \ge \mathbb{P}[\tau_n \le t] \inf_{x\in\mathcal{D}\setminus\mathcal{D}_n} v(x)$, we thus have

$$\mathbb{P}[\tau_n \le t] \le \frac{e^{ct}\mathbb{E}[v(X_0)]}{\inf_{x\in\mathcal{D}\setminus\mathcal{D}_n} v(x)}.$$

Taking the limit $n \to \infty$, we obtain

$$\mathbb{P}[\tau \le t] \le \frac{e^{ct}\mathbb{E}[v(X_0)]}{\liminf_{x \to \partial \mathcal{D}} v(x)} = 0.$$

Thus, taking $t \to \infty$, we prove (a).

Next, since $\tau = +\infty$ a.s., we may consider $(X_t)_{t \ge 0}$ that is now defined for all time. Using Itô's formula,

$$\mathbb{E}[v(X_{\tau_n \wedge t}) - v(X_0)] = \mathbb{E} \int_0^{\tau_n \wedge t} \mathscr{L}v(X_u)du.$$

Note that $\mathbb{E}[v(X_{\tau_n \wedge t}) - v(X_0)] \le b_1$ for some constant b_1. Also note that $\sup_{x \in \mathcal{D}} \mathscr{L}v(x) \le b_2$ for some constant b_2, given assumptions (i)-(iii) and the fact that v is C^2. Using these bounds, plus the following obvious inequality

$$\mathscr{L}v(X_u) \le \mathbf{1}_{\{X_u \in \mathcal{D} \setminus \mathcal{D}_k\}} \sup_{x \in \mathcal{D} \setminus \mathcal{D}_k} \mathscr{L}v(x) + \sup_{x \in \mathcal{D}} \mathscr{L}v(x),$$

we get

$$- \sup_{x \in \mathcal{D} \setminus \mathcal{D}_k} \mathscr{L}v(x) \mathbb{E} \int_0^{\tau_n \wedge t} \mathbf{1}_{\{X_u \in \mathcal{D} \setminus \mathcal{D}_k\}} du \le tb_2 - b_1.$$

Given the proof of (a), we may take the limit $n \to \infty$ (so that $\tau_n \to +\infty$), then apply Fubini's theorem, and then rearrange to obtain

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{P}[X_u \in \mathcal{D} \setminus \mathcal{D}_k] du \le \frac{b_2}{-\sup_{x \in \mathcal{D} \setminus \mathcal{D}_k} \mathscr{L}v(x)}.$$

Taking $k \to \infty$ and using assumption (iii), we obtain

$$\lim_{k \to \infty} \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{P}[X_u \in \mathcal{D} \setminus \mathcal{D}_k] du \le 0.$$

Applying Theorem 3.1 of Khasminskii (2011), there exists a stationary initial distribution for X_0. The process $(X_t)_{t \ge 0}$ augmented with this initial distribution is clearly stationary by definition. □

B.3 Proofs of Corollaries 1-3

Proof of Corollary 1. Start from the construction of S-BSE in Theorem 1, and note that we can make ϵ arbitrarily small such that the boundaries $q_\alpha^L \to \bar{a}/\bar{\rho}$ and $q_\beta^H \to a_e/\bar{\rho}$.

In addition, the limit can be taken such that $\eta^*_\beta \to \eta^*$, its minimal possible level. Hence, an S-BSE can be constructed such that the set of prices q matches $\mathcal{Q}(\eta)$ arbitrarily closely. The result on return variance comes from using (B.4) when $\kappa < 1$ (i.e., when $\eta < \eta^*$) and using (B.5) when $\kappa = 1$ (i.e., when $\eta \geq \eta^*$ and q is at its upper bound). Using the definition of η^* provides the form of $\mathcal{V}$ with the minimum as the lower bound.

To show that an S-BSE can be constructed such that positive probability is placed on all elements of $\mathcal{Q}$ and $\mathcal{V}$, we simply note that a construction exists such that $\sigma_q \neq 0$ on the entirety of $\mathrm{int}(\mathcal{D})$. □

Proof of Corollaries 2-3. These follow from the proof of Theorem 1. □

C Proofs and analysis for Section 4

C.1 Proof of Proposition 2

We proceed by construction. Without loss of generality, let $\mathcal{S} = (0,1)$ and ignore the auxiliary states $\mathcal{X}$, so that the domain of the state variables is $\mathcal{D} = (0,1) \times (0,1)$. The reason we can effectively ignore $\mathcal{X}$ is everything below is that only the drift μ_s depends on x, and x_t is assumed to be a bounded process, thereby introducing no problems of non-stationarity. Recall that $\bar{\rho} := \eta\rho_e + (1-\eta)\rho_h$. By analogy, define $\bar{a} := \eta a_e + (1-\eta)a_h$.

Step 1: Fundamental equilibrium. Let $(\hat{q}^0, \hat{\kappa}^0)$ be the solution to the fundamental equilibrium (which exists by assumption), and let $\eta^0 := \inf\{\eta : \hat{q}^0 \geq a_e/\bar{\rho}\} = \inf\{\eta : \hat{\kappa}^0 \geq 1\}$. By part (v) of Lemma E.1, there exists $\bar{\sigma}_A > 0$ such that, if $\sigma < \bar{\sigma}_A$, then $\eta^0 < 1$. By part (iv) of Lemma E.1, there exists $\bar{\sigma}_B > 0$ such that, if $\sigma < \bar{\sigma}_B$, then $(\hat{q}^0)' > \frac{a_e - a_h}{\bar{\rho}}$ for $\eta \in (0, \eta^0)$. Only to assist with step 9 below, we also denote $\bar{\sigma}_C = \sqrt{\rho_e - \rho_h}\mathbf{1}_{\delta=0} + (+\infty)\mathbf{1}_{\delta>0}$. Assume $\sigma < \min(\bar{\sigma}_A, \bar{\sigma}_B, \bar{\sigma}_C)$. In particular, this implies $\frac{d}{d\eta}[\hat{q}^0 - \bar{a}/\bar{\rho}] > 0$ for $\eta \in (0, \eta^0)$.

Step 2: Two basis functions. We design two "extremal" functions that will assist our construction. First, let φ be a C^2 function with the properties $\varphi(\eta^0) = 0$ and $\varphi' > (\bar{a}/\bar{\rho})' - (a_e/\bar{\rho})' = \frac{a_e - a_h}{\bar{\rho}}[1 - (1-\eta)\frac{\rho_e - \rho_h}{\bar{\rho}}]$ for all η. Define

$$q^0(\eta) := \begin{cases} \hat{q}^0(\eta), & \text{if} \quad \eta < \eta^0; \\ \hat{q}^0(\eta) + \varphi(\eta), & \text{if} \quad \eta \geq \eta^0. \end{cases} \tag{C.1}$$

Note that q^0 is C^∞ except at $\eta = \eta^0$, due to part (vi) of Lemma E.1.

To construct the other basis function, fix some $\epsilon \in (0, \eta^0)$, let $\tilde{\epsilon} \in (\epsilon, \eta^0)$, and define a C^∞ (but necessarily non-analytic) function $\beta : (0,1) \mapsto \mathbb{R}_+$ with the following properties

$$\begin{aligned}
\beta(\epsilon) &= q^0(\epsilon) - \bar{a}(\epsilon)/\bar{\rho}(\epsilon) \\
\beta^{(k)}(\epsilon) &= \frac{d^k}{d\eta^k}[q^0 - \bar{a}(\eta)/\bar{\rho}(\eta)]|_{\eta=\epsilon} \quad \text{for each derivative of order } k \geq 1 \\
\beta'(\eta) &< \frac{d}{d\eta}[q^0 - \bar{a}(\eta)/\bar{\rho}(\eta)] \quad \text{for all} \quad \eta > \epsilon \\
\beta(\eta) &= 0 \quad \text{for all} \quad \eta > \tilde{\epsilon}.
\end{aligned}$$

A particular consequence of $\sigma < \bar{\sigma}_B$ in step 1 is $\frac{d}{d\eta}[q^0 - \bar{a}/\bar{\rho}] > 0$ for $\eta \in (0, \eta^0)$. A consequence of $\varphi' > (\bar{a}/\bar{\rho})' - (a_e/\bar{\rho})'$ is $\frac{d}{d\eta}[q^0 - \bar{a}/\bar{\rho}] > 0$ for $\eta \in (\eta^0, 1)$. Together, these properties imply such a function β exists. Then, we put

$$q^1(\eta) := \begin{cases} \hat{q}^0(\eta), & \text{if} \quad \eta \leq \epsilon; \\ \bar{a}(\eta)/\bar{\rho}(\eta) + \beta(\eta), & \text{if} \quad \eta > \epsilon. \end{cases} \tag{C.2}$$

Note that $\eta^1 := \inf\{\eta : q^1 \geq a_e/\bar{\rho}\} = 1$. By the properties of β and φ, note the following slope results:

$$(q^0)' > (q^1)' \quad \text{on} \quad \eta \in (\epsilon, 1) \tag{C.3}$$

$$(q^0)^{(k)}(\epsilon) = (q^1)^{(k)}(\epsilon) \quad \text{for all derivatives of order } k \geq 0. \tag{C.4}$$

Step 3: Useful monotonicity results. Before continuing, we make the following claims:

$$\frac{\bar{a}}{\bar{\rho}} < q^1 = q^0 < \frac{a_e}{\bar{\rho}}, \quad \text{for} \quad \eta \in (0, \epsilon); \tag{C.5}$$

$$\frac{\bar{a}}{\bar{\rho}} \leq q^1 < q^0 < \frac{a_e}{\bar{\rho}}, \quad \text{for} \quad \eta \in (\epsilon, \eta^0); \tag{C.6}$$

$$\frac{\bar{a}}{\bar{\rho}} = q^1 < \frac{a_e}{\bar{\rho}} < q^0, \quad \text{for} \quad \eta \in (\eta^0, 1). \tag{C.7}$$

All inequalities in relationship (C.5), as well as the third inequality in relationship (C.6), hold by part (ii) of Lemma E.1. The first inequality in relationship (C.6) holds because $\beta \geq 0$, whereas the first equality in relationship (C.7) holds because $\beta = 0$ on that set. The second inequality in relationship (C.6) holds due to (C.3). The second inequality in relationship (C.7) holds by the definition of $\eta^1 = 1$. The second inequality in relationship

(C.7) holds since $q^0(\eta^0) = a_e/\bar{\rho}(\eta^0)$ combined with $(q^0 - a_e/\bar{\rho})' > (q^1 - a_e/\bar{\rho})' > 0$, for $\eta > \eta^0$.

Step 4: Construct candidate (q, κ). We proceed to combine our basis functions according to the following convex combination, where $\alpha \in (0, 1)$ is fixed:

$$\tilde{q}(\eta, s) := (1 - \alpha s)q^0(\eta) + \alpha s q^1(\eta), \quad (\eta, s) \in \mathcal{D} = (0, 1) \times \mathcal{S}. \tag{C.8}$$

For each $s \in \mathcal{S}$, define $\eta^*(s) := \inf\{\eta : \tilde{q}(\eta, s) \geq a_e/\bar{\rho}\}$, which can be shown is strictly increasing.[31] Put

$$q(\eta, s) := \begin{cases} \tilde{q}(\eta, s), & \text{if} \quad \eta < \eta^*(s) \\ a_e/\bar{\rho}(\eta), & \text{if} \quad \eta \geq \eta^*(s) \end{cases} \quad \text{and} \quad \kappa := \frac{\bar{\rho}q - a_h}{a_e - a_h}.$$

By construction, the pair (q, κ) satisfy equation (PO).

Step 5: Properties of (q, κ). Let $\mathcal{D}^* := \{(\eta, s) : \eta \in (\epsilon, \eta^*(s)), s \in \mathcal{S}\}$. On this set, we have $\kappa > \eta$, or equivalently $\bar{\rho}q > \bar{a}$, by (C.6)-(C.7). In fact, κ is bounded away from η on $\mathcal{D}^*$, since $\alpha < 1$ in (C.8). We also have the following derivative conditions on $\mathcal{D}^*$:

$$\partial_s q = \alpha(q^1 - q^0) < 0 \tag{C.9}$$

$$\partial_\eta q = (1 - \alpha s)(q^0)' + \alpha s(q^1)' > 0 \tag{C.10}$$

$$\partial_\eta q < q/(\kappa - \eta). \tag{C.11}$$

Inequality (C.9) holds by (C.6)-(C.7). Inequality (C.10) holds by (C.3) and Assumption 1(ii), which implies $(q^1)' > 0$. Inequality (C.11) is proven as follows. First, note that the function $f(\eta, x) = \frac{(a_e - a_h)x}{\bar{\rho}(\eta)x - \bar{a}(\eta)}$ is strictly decreasing in x on $x > \bar{a}(\eta)/\bar{\rho}(\eta)$. Second, part (i) of Lemma E.1 implies

$$(q^0)' < \frac{(a_e - a_h)q^0}{\bar{\rho}q^0 - \bar{a}} = f(\cdot, q^0).$$

[31] Indeed, note that $\tilde{q}$ is C^2 on $(\eta^0, \eta^1) \times \mathcal{S}$, which implies η^* is C^1. Then, use the fact that η^* is C^1 to differentiate $\tilde{q}(\eta^*(s), s) = a_e/\bar{\rho}(\eta^*(s))$ with respect to s, and use the fact that $\partial_s \tilde{q} = q^1 - q^0$, and finally rearrange to obtain

$$(\eta^*)'(s)\Big[\partial_\eta \tilde{q}(\eta^*(s), s) + \frac{a_e}{\bar{\rho}(\eta^*(s))}\frac{\rho_e - \rho_h}{\bar{\rho}(\eta^*(s))}\Big] = q^0(\eta^*(s)) - q^1(\eta^*(s)).$$

If at any point s, we had $(\eta^*)'(s) = 0$, we would necessarily have $q^0(\eta^*(s)) = q^1(\eta^*(s))$. But this contradicts the fact from (C.6)-(C.7) that $q^0 > q^1$ for all $\eta > \epsilon$, since $\eta^*(s) \geq \eta^0 > \epsilon$ (the fact that $\eta^*(s) \geq \eta^0$ comes from (C.6), which shows that $\tilde{q}(\eta, s) < a_e/\bar{\rho}(\eta)$ on $(\epsilon, \eta^0) \times \mathcal{S}$). Thus, $(\eta^*)'(s) \neq 0$ for all s. We can also rule out $(\eta^*)'(s) < 0$ by the fact that $\eta^*(0+) = \eta^0$ and $\eta^*(s) \geq \eta^0$ for all s. Thus, $(\eta^*)'(s) > 0$ for all s.

Given (C.3), we thus have $\partial_\eta q < f(\cdot, q^0)$ for any value of s. Finally, since f is decreasing in its second argument, and $q < q^0$ on $\mathcal{D}^*$, we have $\partial_\eta q < f(\cdot, q)$, which proves the claim.

We remark on one additional smoothness property that holds at $\eta = \epsilon$, due to condition (C.4):

$$\partial_\eta^{(k)} q(\epsilon, s) = (q^0)^{(k)}(\epsilon) \quad \forall s, \quad \text{for all derivatives of order } k \geq 0. \tag{C.12}$$

Step 6: Construct candidate σ_s. Consider solving the following problem.

$$\begin{aligned} \textit{Problem:} \quad & \text{for each } (\eta, s) \in \mathcal{D}^*, \text{ solve for } y \text{ in the equation} \\ & y(\partial_s \log q)^2 = G, \end{aligned} \tag{C.13}$$

where

$$G := \frac{\eta(1-\eta)}{\kappa - \eta} \frac{a_e - a_h}{q} (1 - (\kappa - \eta)\partial_\eta \log q)^2 - \sigma^2.$$

Note that G is bounded, as κ is bounded away from η (step 5). Checking boundedness of the solution y thus boils down to checking $\partial_s q$ at the boundaries of $\mathcal{D}^*$. By (C.9), as $s \to 0$ or $s \to 1$, $\partial_s q \not\to 0$, so y remains bounded. To check the result as $\eta \to \epsilon$, we first claim that $\lim_{\eta \searrow \epsilon} \partial_\eta^{(k)} G = 0$ for all derivatives of order $k \geq 0$. This is a consequence of parts (i) and (vi) of Lemma E.1, whereby $\partial_\eta^{(k)} G = 0$ for all $k \geq 0$ on $\eta < \epsilon$, combined with result (C.12). Since we also have $\partial_s q \to 0$, we apply L'Hôpital's rule twice to compute $\lim_{\eta \searrow \epsilon} G/(\partial_s \log q)^2 = 0$, noting both times that $\partial_{s\eta} \log q = \frac{\alpha}{q}[(q^1)' - (q^0)'] < 0$ is non-zero. Therefore, the solution $y = G/(\partial_s \log q)^2$ is bounded on $\mathcal{D}^*$.

Clearly, $\sqrt{y}$ will be a real number if and only if $G \geq 0$. To prove $G \geq 0$, note that $\lim_{s \searrow 0} G = 0$, meaning it suffices to prove $\partial_s G \geq 0$. Differentiating G, we get

$$\begin{aligned} \frac{\partial_s G}{\eta(1-\eta)} = -\frac{a_e - a_h}{(\kappa - \eta)q}(1 - (\kappa - \eta)\partial_\eta \log q)\Big[(1 - (\kappa - \eta)\partial_\eta \log q)\Big(\frac{\partial_s \kappa}{\kappa - \eta} + \frac{\partial_s q}{q}\Big) \\ + 2\frac{(\kappa - \eta)\bar{a}}{\bar{\rho} q - \bar{a}}(\partial_s \log q)(\partial_\eta \log q) + 2(\kappa - \eta)\alpha \frac{(q^1)' - (q^0)'}{q}\Big]. \end{aligned}$$

By properties (C.9)-(C.11), and the fact that $\mathrm{sgn}(\partial_s \kappa) = \mathrm{sgn}(\partial_s q)$, we prove $\partial_s G > 0$ on $\mathcal{D}^*$. So not only is $\sqrt{y}$ real, it is non-zero.

We set σ_s as follows:

$$\sigma_s(\eta,s) := \begin{cases} \sqrt{y(\eta,s)}, & \text{if} \quad (\eta,s) \in \mathcal{D}^*; \\ \sqrt{y(\epsilon+,s)} = 0, & \text{if} \quad (\eta,s) \in \{(\eta,s) : \eta \in (0,\epsilon),\, s \in \mathcal{S}\}; \\ \sqrt{y(\eta^*(s)-,s)}, & \text{if} \quad (\eta,s) \in \{(\eta,s) : \eta > \eta^*(s),\, s \in \mathcal{S}\}. \end{cases} \tag{C.14}$$

In passing, we note that we have also shown that $\sigma_s > 0$ on a positive-measure set, as required in a sunspot equilibrium.

Step 7: Verify equation (22) is satisfied. By the construction of σ_s, equation (22) is satisfied on $\mathcal{D}^*$. On $\{(\eta,s) : \eta \in (0,\epsilon),\, s \in \mathcal{S}\}$, recall $\partial_s q = 0$, so (22) holds by property (i) of Lemma E.1. On $\{(\eta,s) : \eta > \eta^*(s),\, s \in \mathcal{S}\}$, recall $\kappa = 1$, so (22) is satisfied if and only if the second term inside the minimum is non-negative. Substituting $\kappa = 1$ and $q = a_e/\bar{\rho}$, hence $\partial_s q = 0$, into this term shows the non-negativity requirement is

$$\sigma^2 \leq \eta\bar{\rho}\frac{a_e - a_h}{a_e}(1 + (1-\eta)\partial_\eta \log\bar{\rho})^2 \quad \text{for} \quad \eta > \eta^*(s),\, s \in \mathcal{S}. \tag{C.15}$$

On the other hand, property (v) of Lemma E.1, combined with the fact that $\eta^*(s)$ is increasing, imply

$$\eta^*(s) \geq \frac{\rho_h}{\rho_e}\Big(\frac{1 - a_h/a_e}{\sigma^2}\rho_e - 1 + \frac{\rho_h}{\rho_e}\Big)^{-1}, \quad \forall s \in \mathcal{S}. \tag{C.16}$$

Straightforward algebra demonstrates that (C.15) and (C.16) are equivalent, proving (22) holds.

Step 8: Finish equilibrium construction. Having determined q, κ, and σ_s, we define μ_η and σ_η by (11)-(12). It remains to determine μ_s. We will pick $\mu_s(\eta,s) = m(\eta,s)$, where m is a C^2 function with the following properties: $\partial_s m < 0$, and for some $0 \leq s^0 < s^1 \leq 1$ thresholds,

$$(\text{if } s^0 > 0) \qquad \inf_{\eta\in(0,1)} \lim_{s\searrow s^0} (s - s^0)m(\eta,s) = +\infty \tag{C.17}$$

$$(\text{if } s^0 = 0) \qquad \inf_{\eta\in(0,1)} \lim_{s\searrow s^0} m(\eta,s) > 0 \tag{C.18}$$

$$\sup_{\eta\in(0,1)} \lim_{s\nearrow s^1} (s^1 - s)m(\eta,s) = -\infty. \tag{C.19}$$

Step 9: Verify stationarity. Finally, we should demonstrate the time-paths $(\eta_t, s_t)_{t\geq 0}$ remain in $\mathcal{D}$ almost-surely and admit a stationary distribution. This step is very similar to Theorem 1 and is therefore omitted. $\square$.

C.2 Proofs of Propositions 3-4

PROOF OF PROPOSITION 3. Fix any $\Sigma^* > 0$. The proof is a simple consequence of the fact that σ_q must be unbounded as κ approaches η, which is as q approaches the worst-case price q^1. We fill in the technical details below.

We construct a sequence of equilibria—indexed by $(\alpha, \epsilon, \zeta)$—as follows. Recall the capital price construction in Proposition 2:

$$q = (1 - \alpha s)q^0 + \alpha s q^1, \quad \text{when} \quad \kappa < 1,$$

where $\alpha < 1$ is a parameter, q^0 is the fundamental equilibrium price, and

$$q^1 = \begin{cases} q^0, & \text{if } \eta < \epsilon; \\ \bar{a}/\bar{\rho} + \beta, & \text{if } \eta \in (\epsilon, \tilde{\epsilon}); \\ \bar{a}/\bar{\rho}, & \text{if } \eta > \tilde{\epsilon}. \end{cases}$$

The function β is a positive mollifier that vanishes uniformly as $\epsilon, \tilde{\epsilon} \to 0$. We set $\tilde{\epsilon} = \epsilon(1+\epsilon)$. Based on the discussion in the text, we may choose μ_s such that equilibrium concentrates on any particular value of s. Thus, pick μ_s such that $s_t \geq \zeta$ almost-surely. Clearly, the choice of μ_s depends on α and ϵ, but such a choice can always be made for any parameters.

Let $p_{\text{low}} > 0$, $p_{\text{high}} > 0$ be given with $p_{\text{low}} + p_{\text{high}} < 1$. First, note that there exist α^*, ζ^*, and ϵ^* such that $\mathbb{P}[\eta_t \leq \tilde{\epsilon} \cap \kappa_t < 1] < p_{\text{low}}$ and $\mathbb{P}[\eta_t \geq 1 - \tilde{\epsilon} \cap \kappa_t < 1] < p_{\text{high}}$ for all $\alpha > \alpha^*$, $\zeta > \zeta^*$, and $\epsilon < \epsilon^*$. This is a consequence of the fact that in any stationary distribution, we have $\lim_{x \to 0} \mathbb{P}[\eta_t < x] = \lim_{x \to 1} \mathbb{P}[\eta_t > x] = 0$ and the fact that $\lim_{\alpha \to 1} \lim_{s \to 1} \kappa(\eta, s) < 1$ for all η.

At this point, fix such an $\epsilon < \epsilon^*$. Let a constant $M > 0$ be given satisfying

$$M \leq (1 - p_{\text{low}} - p_{\text{high}}) \frac{(a_e - a_h)^2}{a_e/\rho_h} \frac{\tilde{\epsilon}(1-\tilde{\epsilon})}{\Sigma^*}. \tag{C.20}$$

Note that

$$\lim_{\alpha \to 1} \lim_{s \to 1} \sup_{\eta \in (\tilde{\epsilon}, 1-\tilde{\epsilon})} \Big| q(\eta, s) - \bar{a}(\eta)/\bar{\rho}(\eta) \Big| = 0.$$

Consequently, we may pick $\alpha > \alpha^*$ close enough to 1 and $\zeta > \zeta^*$ close enough to 1 such that

$$\sup_{s \in (\zeta,1)} \sup_{\eta \in (\tilde{\epsilon}, 1-\tilde{\epsilon})} \Big| q(\eta, s) - \bar{a}(\eta)/\bar{\rho}(\eta) \Big| \leq M.$$

Finally, using equation (22) and substituting $\kappa < 1$ from (PO), we have $|\sigma\binom{1}{0} + \sigma_q|^2 = \frac{(a_e - a_h)^2}{q} \frac{\eta(1-\eta)}{\bar{\rho}q - \bar{a}}$. Note also that $q \leq a_e/\rho_h$ is an upper bound. Then,

$$\mathbb{E}[|\sigma\tbinom{1}{0} + \sigma_{q,t}|^2] > (1 - p_{\text{low}} - p_{\text{high}}) \frac{(a_e - a_h)^2}{a_e/\rho_h} \frac{\tilde{\epsilon}(1-\tilde{\epsilon})}{M}.$$

Using (C.20), we obtain $\mathbb{E}[|\sigma\binom{1}{0} + \sigma_{q,t}|^2] > \Sigma^*$. $\square$

Proof of Proposition 4. First, we prove that $|\sigma_R|$ is increasing in s. From (22), we obtain $|\sigma_R|^2 = \frac{(a_e - a_h)^2}{q} \frac{\eta(1-\eta)}{\bar{\rho}q - \bar{a}}$ on $\{\kappa < 1\}$. Differentiating with respect to s, we obtain

$$\partial_s |\sigma_R|^2 = -\eta(1-\eta) \frac{(a_e - a_h)^2}{q(\bar{\rho}q - \bar{a})} \left[\frac{1}{q} + \frac{\bar{\rho}}{\bar{\rho}q - \bar{a}} \right] \partial_s q > 0,$$

since $\partial_s q = \alpha(q^1 - q^0) < 0$ by (C.9).

Next, we show that $|\binom{1}{0} \cdot \sigma_R|$ is decreasing in s. Revisiting the proof of Proposition 2, we compute on $\{\kappa < 1\}$ and for each $\eta > \epsilon$,

$$\partial_s[(\kappa - \eta)\partial_\eta \log q] = \alpha \Big[(\kappa - \eta) \frac{(q^1)' - (q^0)'}{q} + \frac{\bar{a}(q^1 - q^0)}{(a_e - a_h)q^2} \partial_\eta q \Big] < 0.$$

The inequality uses (C.3) to say $(q^1)' - (q^0)' < 0$, and (C.6)-(C.7) to say $q^1 - q^0 < 0$, and (C.10) to say $\partial_\eta q > 0$. Therefore, $(1 - (\kappa - \eta)\partial_\eta \log q)^{-1}$ is decreasing in s on $\{\kappa < 1\}$ for each $\eta > \epsilon$. Since q and κ are independent of s on $\{\eta < \epsilon\}$, this proves $(1 - (\kappa - \eta)\partial_\eta \log q)^{-1}$ is weakly decreasing in s on $\{\kappa < 1\}$. Using $|\binom{1}{0} \cdot \sigma_R| = \frac{\sigma}{1 - (\kappa - \eta)\partial_\eta \log q}$, we obtain the result.

Using the two claims just proved, we see that $|\binom{0}{1} \cdot \sigma_R|$ is increasing in s on $\{\kappa < 1\}$, due to the identity $|\sigma_R|^2 = |\binom{0}{1} \cdot \sigma_R|^2 + |\binom{1}{0} \cdot \sigma_R|^2$. For the same reason, we have $|\binom{0}{1} \cdot \sigma_R| / |\sigma_R|$ increasing in s on $\{\kappa < 1\}$. $\square$

C.3 Model with jumps in Section 4.4

Recall that our jumps ℓ_q are assumed to occur randomly but have a known size, given observables. Therefore, optimal portfolio conditions are

$$\frac{a_e}{q} + g + \mu_q + \sigma\binom{1}{0} \cdot \sigma_q - r = \frac{\kappa}{\eta}|\sigma_R|^2 + \frac{\lambda \ell_q}{1 - \frac{\kappa}{\eta}\ell_q}$$
$$\frac{a_h}{q} + g + \mu_q + \sigma\binom{1}{0} \cdot \sigma_q - r \le \frac{1-\kappa}{1-\eta}|\sigma_R|^2 + \frac{\lambda \ell_q}{1 - \frac{1-\kappa}{1-\eta}\ell_q}.$$

Combining these two equations, we obtain (RBJ).

We can determine the other equilibrium objects similarly to before. The riskless rate is given by

$$r = \frac{\kappa a_e + (1-\kappa)a_h}{q} + g + \mu_q + \sigma\binom{1}{0} \cdot \sigma_q - \Big(\frac{\kappa^2}{\eta} + \frac{(1-\kappa)^2}{1-\eta}\Big)|\sigma_R|^2 - \lambda \ell_q \Big(\frac{\kappa}{1 - \frac{\kappa}{\eta}\ell_q} + \frac{1-\kappa}{1 - \frac{1-\kappa}{1-\eta}\ell_q}\Big).$$

The dynamics of η are now given by $d\eta_t = \mu_{\eta,t-}dt + \sigma_{\eta,t-} \cdot dZ_t - \ell_{\eta,t-}dJ_t$, where

$$\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + (\kappa - 2\eta\kappa + \eta^2)\frac{\kappa - \eta}{\eta(1-\eta)}|\sigma_R|^2 + \delta(\nu - \eta) + \frac{(\kappa - \eta)\lambda\ell_q}{\big(1 - \frac{\kappa}{\eta}\ell_q\big)\big(1 - \frac{1-\kappa}{1-\eta}\ell_q\big)}$$
$$\sigma_\eta = (\kappa - \eta)\sigma_R.$$

The wealth share jump ℓ_η is derived by using knowledge of the jump size in q and noting that agents' portfolios (capital and bonds) are predetermined:[32]

$$\ell_\eta = (\kappa - \eta)\frac{\ell_q}{1 - \ell_q}.$$

For a valid equilibrium, jumps cannot be so large as to send experts into bankruptcy, nor can they induce households' leverage to exceed experts' (as this would contradict (RBJ)). It turns out the no-bankruptcy condition, which says $\ell_q < \kappa/\eta$, is automatically satisfied given (RBJ) holds; intuitively, experts would never take so much risk that their wealth is wiped out. The other requirement, that jumps not send the economy into a region in

[32]The derivation is as follows. Let variables with hats, e.g., "$\hat{x}$", denote post-jump variables. Note $\hat{N}_e = \hat{q}\hat{K}\kappa - B$ and $\hat{N}_h = \hat{q}\hat{K}(1-\kappa) + B$, where B is expert borrowing (and household lending, by bond market clearing). Then, $\hat{\eta} = \hat{N}_e/(\hat{q}\hat{K}) = \kappa - B/(\hat{q}\hat{K})$ and by similar logic the pre-jump wealth share is $\eta = \kappa - B/qK$. Thus, $\ell_\eta = \eta - \hat{\eta} = B[1/(\hat{q}\hat{K}) - 1/(qK)] = qK(\kappa - \eta)[1/(\hat{q}\hat{K}) - 1/(qK)]$. Using the fact that $\hat{K} = K$ and the definition $\ell_q := 1 - \hat{q}/q$, we arrive at $\ell_\eta = (\kappa - \eta)[(1-\ell_q)^{-1} - 1]$. This derivation assumes the presumably risk-free bond price does not jump when capital prices jump. Conceptually, there is no reason why this needs to be true, but it preserves its risk-free conjecture. If bond prices are allowed to jump at the same time, we would find different expressions.

which $\eta \leq \kappa$, can be stated as

$$\bar{\rho}(\hat{\eta})(1-\ell_q)q > (a_e - a_h)\hat{\eta} + a_h, \tag{C.21}$$

where $\hat{\eta} := \eta - (\kappa - \eta)\frac{\ell_q}{1-\ell_q}$ is the post-jump expert wealth share. Although it is obvious, (RBJ) implies another bound on ℓ_q that arises because of $|\sigma_R| \geq 0$, which is

$$\frac{a_e - a_h}{q} \geq \frac{\kappa - \eta}{\eta(1-\eta)}\Big[\frac{\lambda \ell_q^2}{\left(1 - \frac{\kappa}{\eta}\ell_q\right)\left(1 - \frac{1-\kappa}{1-\eta}\ell_q\right)}\Big]. \tag{C.22}$$

Condition (C.22) evaluated at equality implies that all risk is jump risk. With these equations in hand, we describe our simulation procedure.

Step 0. Given (η, q) solve for $\kappa(\eta, q)$ from (PO).

Step 1. Solve for the upper bound of $\ell_q(\eta, q)$ using (C.21)-(C.22).

Note that, fixing (η, q), the RHS of (C.22) is strictly increasing in ℓ_q when $\ell_q \in \left(0, \frac{\eta}{\kappa}\right)$ while the LHS is constant. Moreover, the inequality is satisfied for $\ell_q = 0$ and violated as $\ell_q \to \frac{\eta}{\kappa}$. Hence, this condition defines an upper bound $\ell_q^A(\eta, q)$, which can be solved by a bisection procedure.

Next, after some algebra, we can write condition (C.21) as

$$(1-\ell_q)^2 - (1-\ell_q) + \underbrace{\frac{(a_e - a_h)(\kappa - \eta)}{\bar{\rho}(\eta)q + q(\rho_e - \rho_h)(\kappa - \eta)}}_{:=\varphi(\eta, q)} > 0.$$

It is straightforward to notice that the condition holds for any $\ell_q \in (0, 1)$ if $\varphi(\eta, q) \geq 1/4$. When $\varphi(\eta, q) < 1/4$, then the condition holds for $\ell_q \in (0, \ell_q^{B,low}) \cup (\ell_q^{B,high}, 1)$, where

$$1 - \ell_q^{B,high} = \frac{1}{2}\left(1 - \sqrt{1 - 4\varphi}\right) \quad \text{and} \quad 1 - \ell_q^{B,low} = \frac{1}{2}\left(1 + \sqrt{1 - 4\varphi}\right).$$

Define

$$\ell_q^B := \mathbf{1}_{\{\varphi \geq 1/4\}} + \ell_q^{B,low}\mathbf{1}_{\{\varphi < 1/4\}}.$$

Then, an upper bound that ensures all required inequalities are satisfied is

$$\ell_q^{\max}(\eta, q) := \min\{\ell_q^A(\eta, q), \ell_q^B(\eta, q)\}.$$

Step 2. Choose a sub-region within the domain $\mathcal{D} := \{(\eta, q) : 0 < \eta < 1 \text{ and } \eta a_e + (1 - \eta)a_h < q\bar{\rho}(\eta) \leq a_e\}$ that is away from the upper and lower boundaries. For example, in our numerical exercise, we choose the sub-region $\mathcal{D}^\circ := \{(\eta, q) : \kappa < 0.98 \text{ and } \kappa > \eta + 0.02\}$. On $\mathcal{D} \backslash \mathcal{D}^\circ$, we will set $\ell_q = 0$ and choose μ_q to ensure the economy never escapes $\mathcal{D}$. In fact, we can choose μ_q in a way that the boundary of $\mathcal{D}^\circ$ acts arbitrarily close to a reflecting boundary, which is what we have done in for Figure 9. Pick an arbitrary function $\ell_q(\eta, q) \in [0, \ell_q^{\max}(\eta, q))$ and an arbitrary μ_q for the set $\mathcal{D}^\circ$.

Step 3. Use risk-balance condition (RBJ) to solve for $|\sigma_R|^2$. For each (η, q), assign $\gamma(\eta, q)$ fraction of the variance to the fundamental Brownian shock, and $1 - \gamma(\eta, q)$ to the sunspot Brownian shock. In constructing Figure 9, we set $\gamma \equiv 1$. Then, solve for other equilibrium objects from the equations above.

Online Appendix 2 (not for publication):

Rational Sentiments and Financial Frictions

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D Model extensions and further analyses

D.1 Beliefs about disaster states

In this section, we outline a richer class of W-BSE when $\sigma = 0$. The entire set of W-BSEs studied here will be indexed by agents' beliefs about the "tail scenario" in the economy, i.e., what happens when experts are severely undercapitalized.

Mathematically, recall that we previously have assumed $\kappa(0) = 0$; in other words, experts fully deleverage as their wealth vanishes. Some intuitive refinements like a small amount of idiosyncratic risk (Appendix D.2) or a small amount of commitment frictions (Appendix D.3) can justify the assumption $\kappa(0) = 0$. However, strictly speaking, $\kappa(0) = 0$ turns out to not be necessary without these refinements, and it will be interesting to relax this assumption.

Consider any $\kappa_0 \in (0, 1)$ and put $\kappa(0) = \kappa_0$. We will call κ_0 the *disaster belief* in the economy. The sunspot equilibrium is similar to Proposition 1, with the generalization that the boundary condition to the ODE (15) is now $\kappa(0) = \kappa_0$ rather than $\kappa(0) = 0$.[33]

Proposition D.1. *For $\sigma = 0$ and fixed tail belief $\kappa_0 \in (0, 1)$, there exists a W-BSE, with $\sigma_q(\eta) \neq 0$ on a positive measure subset of $(0, 1)$. As $\kappa_0 \to 0$, this equilibrium converges to the W-BSE of Proposition 1. As $\kappa_0 \to 1$, the equilibrium converges to the FE of Lemma 1.*

Based on Proposition D.1, proved at the end of this section, one can view both the W-BSE and the FE as outcomes of coordination on experts' deleveraging. If experts never sell any capital, there can be no price volatility, with $\sigma_q = 0$ at all times. If agents expect

[33] As in footnote 17, there is a closed-form solution when $\rho_h = \rho_e$, which is

$$q(\eta) = \frac{1}{\rho}\Big[(a_e - a_h)\eta + a_h + \sqrt{((a_e - a_h)\eta + a_h)^2 - a_h^2 + (a_e - a_h)^2\kappa_0^2}\Big], \quad \text{for} \quad \eta < \eta^* = \frac{1}{2}\frac{a_e - a_h}{a_e}(1 - \kappa_0^2).$$

As κ_0 decreases, the slope $q'(\eta)$ increases, consistent with the idea that pessimism about the disaster state raises the sensitivity of equilibrium to sunspot shocks away from disaster. Clearly, this solution converges to the W-BSE solution in footnote 17 as $\kappa_0 \to 0$, and to the FE solution a_e/ρ as $\kappa_0 \to 1$.

$\kappa_0 = 0$, which translates to full deleveraging and large capital fire sales, then the W-BSE prevails. But for any $\kappa_0 \in (0,1)$, an intermediate sunspot equilibrium will prevail, with a self-fulfilling amount of expert deleveraging and associated price dynamics. In this simple way, the boundary condition $\kappa_0 \in [0,1]$ spans an entire range of sunspot equilibria from more to less volatile. An illustration is in Figure D.1.[34]

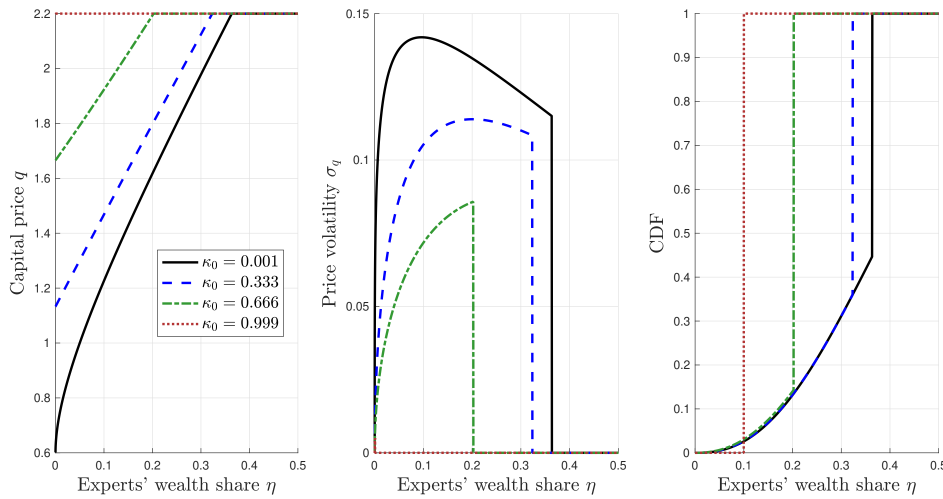

Figure D.1: Capital price q, volatility σ_q, and stationary CDFs of η for different levels of disaster belief κ_0. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$. OLG parameters (for the CDF): $\nu = 0.1$ and $\delta = 0.04$.

This result provides a clear illustration of the central property that the degree of capital fire sales is indeterminate in these models. Intuitively, greater optimism about other experts' ability to retain capital in the tail scenario induces smaller capital fire sales in response to sunspot shocks, which keeps volatility low, asset prices high, and justifies the optimism.

Proof of Proposition D.1. In the first step, we prove existence of an equilibrium for fixed $\kappa_0 \in (0,1)$. In the second step, we take the limits as $\kappa_0 \to 0$ and $\kappa_0 \to 1$.

Step 1: Existence. Let $F(x,y) := \frac{a_e - a_h}{y\bar{\rho}(x) - xa_e - (1-x)a_h} y$. Fix $\epsilon > 0$. Consider the initial value problem $y' = F(x,y)$, with $y(0) = (\kappa_0 a_e + (1-\kappa_0)a_h)/\rho_h$. As discussed in the text, $y'(0+)$ is bounded, which is enough to ensure that F is bounded and uniformly Lipschitz on $\mathcal{R} := \{(x,y) : 0 < x < 1,\, xa_e + (1-x)a_h < y\bar{\rho}(x)\}$. Thus, the standard Picard-Lidelöf theorem implies that there exists a unique solution q^* to this initial value problem, for $\eta \in (0,b)$, some b. Standard continuation arguments can be used to show that either (i)

[34] This result is also convenient in some numerical situations. Since the W-BSE is just the limit of equilibria as $\kappa_0 \to 0$, we can construct an approximate numerical solution with κ_0 very small (but not quite 0).

$b = 1$, (ii) $q^*(\eta)$ is unbounded as $\eta \to b$, or (iii) b satisfies $ba_e + (1-b)a_h = q^*(b)\bar{\rho}(b)$. If case (ii) is true, since $F > 0$ on $\mathcal{R}$, we will in fact have $q^*(b-) = +\infty$. Case (iii) is ruled out by the fact that $F(b-, q^*(b-)) = +\infty$. We are left with cases (i) or (ii).

In case (i), we will set $\eta^* = \inf\{\eta \in (0,1) : q^*(\eta) = a_e/\bar{\rho}(\eta)\}$, with the convention that $\eta^* = 1$ if this set is empty. Note that $\eta^* < 1$ in this case: otherwise $q^*(1-)\bar{\rho}(1-) < a_e$, which implies $F(1-, q^*(1-)) < 0$, which by continuity of q^* and F implies an $\eta^\circ \in (0,1)$ such that $\eta^\circ a_e + (1-\eta^\circ)a_h = q^*(\eta^\circ)$, which was just ruled out (case (iii)). In case (ii), we will set $\eta^* = \inf\{\eta \in (0,b) : q^*(\eta) = a_e/\bar{\rho}(\eta)\}$, with the convention that $\eta^* = 1$ if this set is empty. Note that we also clearly have $\eta^* < b < 1$ in this case.

Finally, set $q(\eta) = \mathbf{1}_{\eta<\eta^*} q^*(\eta) + \mathbf{1}_{\eta \geq \eta^*} a_e/\bar{\rho}(\eta)$. This function satisfies $q' = F(\eta, q)$ on $(0, \eta^*)$, $q(0) = (\kappa_0 a_e + (1-\kappa_0)a_h)/\rho_h$, and $q(\eta^*) = a_e/\bar{\rho}(\eta)$. Thus, we have found a solution to the capital price satisfying all the desired relations. As discussed in the text, finding such a capital price is enough to prove that a Markov sunspot equilibrium exists.

Since equation (16) implies $\sigma_q^2 > 0$ on $(0, \eta^*)$, in order to establish $\sigma_q(\eta) \neq 0$ on a positive measure subset, it suffices to show that $\eta^* > 0$. But this is automatically implied by the boundary condition $q(0) = (\kappa_0 a_e + (1-\kappa_0)a_h)/\rho_h < a_e/\rho_h$ for $\kappa_0 < 1$, coupled with the continuity of the solution $q(\eta)$.

Step 2: W-BSE and FE as limiting equilibria. For each initial condition $\kappa(0) = \kappa_0$, let $(q_{\kappa_0}, \eta^*_{\kappa_0})$ be the associated equilibrium capital price and misallocation threshold (at which point households begin purchasing capital).

Define the candidate solution for the W-BSE, $(q_0, \eta_0^*) := \lim_{\kappa_0 \to 0}(q_{\kappa_0}, \eta^*_{\kappa_0})$. It suffices to show that q_0 satisfies (i) $q_0' = F(\eta, q_0)$ on $(0, \eta_0^*)$, (ii) $q_0(0) = a_h/\rho_h$, and (iii) $q_0(\eta_0^*) = a_e/\bar{\rho}(\eta_0^*)$. Write the integral version of the ODE:

$$q_{\kappa_0}(\eta) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h} + \int_0^\eta F(x, q_{\kappa_0}(x))dx.$$

Next, we claim that $q_{\kappa_0}(x)$ is weakly increasing in κ_0, for each x. Indeed, $q_{\kappa_0}(0)$ is strictly increasing in κ_0. By continuity, we may consider $x^* := \inf\{x : q_{\tilde{\kappa}_0}(x) = q_{\kappa_0}(x)\}$ for some $\tilde{\kappa}_0 > \kappa_0$. In that case, since F does not depend on $\tilde{\kappa}_0$ or κ_0, we have $q_{\tilde{\kappa}_0}(x) = q_{\kappa_0}(x)$ for all $x \geq x^*$. This proves $q_{\tilde{\kappa}_0}(x) \geq q_{\kappa_0}(x)$ for all x. Combine this with the fact that $\partial_q F < 0$ to see that $\{F(x, q_{\kappa_0}(x)) : \kappa_0 \in (0,1)\}$ is a sequence which is monotonically (weakly) decreasing in κ_0, for each x. Thus, by the monotone convergence theorem,

$$q_0(\eta) = \frac{a_h}{\rho_h} + \int_0^\eta F(x, q_0(x))dx,$$

which proves (i), by differentiating, and (ii), by substituting $\eta = 0$. Similarly,

$$\begin{aligned} q_{\kappa_0}(\eta^*_{\kappa_0}) &= \frac{a_e}{\bar{\rho}(\eta^*_{\kappa_0})} \\ \xrightarrow{\kappa_0 \to 0} q_0(\eta^*_0) &= \frac{a_e}{\bar{\rho}(\eta^*_0)}, \end{aligned}$$

which proves (iii).

Define the candidate solution for the FE, $(q_1, \eta^*_1) := \lim_{\kappa_0 \to 1}(q_{\kappa_0}, \eta^*_{\kappa_0})$. It suffices to show that $\eta^*_1 = 0$, so that $q_1(\eta) = a_e/\bar{\rho}(\eta)$ for all η. Note that $q_{\kappa_0}(0) \to a_e/\rho_h$ as $\kappa_0 \to 1$. By continuity of $(q_{\kappa_0}, \eta^*_{\kappa_0})$ in κ_0, we also have $q_{\kappa_0}(0) \to q_1(0)$ as $\kappa_0 \to 1$. Thus, $q_1(0) = a_e/\rho_h$. By the definition of $\eta^*_1 = \inf\{\eta : q(\eta) = a_e/\bar{\rho}(\eta)\}$, we must have $\eta^*_1 = 0$. □

D.2 Idiosyncratic uncertainty

Here, we add idiosyncratic risk to capital. Doing so raises 3 substantive points: (1) small idiosyncratic uncertainty can provide an equilibrium refinement, by selecting equilibria with the property $\lim_{\eta \to 0} \kappa = 0$; (2) large idiosyncratic uncertainty eliminates sunspot equilibria where η is the sole state variable (i.e., where sunspot shocks are iid); (3) idiosyncratic uncertainty allows us to study, in a non-trivial way, the stability properties of the "deterministic steady state" of our model.

Setting. In addition to the model assumptions listed in Section 1, individual capital now evolves as

$$dk_{i,t} = k_{i,t}[g dt + \tilde{\sigma} d\tilde{B}_{i,t}], \tag{D.1}$$

where $(\tilde{B}_i)_{i \in [0,1]}$ is a continuum of independent Brownian motions. Agents with indexes $i \in [0, I]$ are experts, and those with $i \in [I, 1]$ are households. As in Section 1, the aggregate stock of capital $K_t := \int_0^1 k_{i,t} di$ grows deterministically at rate g (no aggregate shocks).

As before, suppose Z is a one-dimensional Brownian motion (a sunspot shock), independent of all $\tilde{B}_i$. Conjecture

$$dq_t = q_t[\mu_{q,t} dt + \sigma_{q,t} dZ_t].$$

We will focus on Markov equilibria in which η is the sole state variable. A *fundamental equilibrium* features $\sigma_q \equiv 0$. A *sunspot equilibrium* features σ_q which is not identically zero.

Small uncertainty as equilibrium refinement. The first result in this environment is that *any* equilibrium (even one with additional state variables beyond η) must feature full deleveraging by experts, as they become poor, simply as a consequence of portfolio optimality. To see this, note that risk balance condition (RB), the combination of expert and household capital FOCs, is now modified to read

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}(\tilde{\sigma}^2 + \sigma_q^2)\right]. \tag{D.2}$$

Note that $a_e - a_h > 0$ and $\tilde{\sigma}^2 + \sigma_q^2 > 0$. Thus, as $\eta \to 0$, we must have $\kappa \to 0$. Since this holds for any arbitrarily small $\tilde{\sigma}$, we conclude that the equilibria with disaster beliefs $\kappa_0 > 0$ (see Section D.1) are not robust.

Lemma D.1. *Any equilibrium with* $\tilde{\sigma} > 0$ *has the property* $\lim_{\eta\to 0}\kappa = 0$.

Large uncertainty eliminates iid sunspots. In Section 2.2, we have demonstrated how sunspot equilibria with η as the sole state variable are incompatible with the presence of exogenous aggregate fundamental risk. Here, we show that the conclusion is similar if the exogenous risk is idiosyncratic rather than aggregate.

Even with idiosyncratic risk $\tilde{\sigma}$, one may follow the same analysis as Section 2.1 to show that equation (15) still determines q if $\sigma_q \neq 0$. In other words, the candidate sunspot equilibrium of this model has a solution (q, κ), both as functions of η, which are independent of the amount of idiosyncratic risk $\tilde{\sigma}$ (i.e., the same as in the W-BSE). Denote $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$ the boundary point where households begin managing capital. This is also independent of $\tilde{\sigma}$.

Next, use equation (D.2) to solve for σ_q, given the solutions (q, κ). We get

$$\sigma_q^2 = -\tilde{\sigma}^2 + \frac{\eta(1-\eta)}{\kappa - \eta}\frac{a_e - a_h}{q}, \quad \text{if} \quad \kappa < 1.$$

Since $\sigma_q^2 \geq 0$ is required, an immediate consequence is that $\tilde{\sigma}$ high enough eliminates the existence of any sunspot volatility. We collect these results in the following lemma.

Lemma D.2. *Let* (q, κ, η^*) *be given by the W-BSE of Proposition 1. If capital has idiosyncratic risk* $\tilde{\sigma}$, *and* $\tilde{\sigma}^2 \geq \sup_{\eta < \eta^*} \frac{\eta(1-\eta)}{\kappa(\eta)-\eta}\frac{a_e - a_h}{q(\eta)}$, *any Markov equilibrium in* η *requires* $\sigma_q = 0$.

Intuitively, there is a trade-off between endogenous volatility σ_q and exogenous volatility $\tilde{\sigma}$. With higher idiosyncratic volatility $\tilde{\sigma}$, amplification of the aggregate sunspot shock is necessarily reduced. To understand this, consider Merton's optimal capital portfolio when there is only idiosyncratic volatility

$$\frac{qk_j}{n_j} = \frac{a_j/q + g - r}{\tilde{\sigma}^2}, \quad j \in \{e, h\}.$$

As $\tilde{\sigma}$ increases the optimal capital demand becomes more inelastic to changes in the capital price q. Thus, for a given shift in the wealth distribution η and change in capital price q, the amount of capital that changes hands between experts and households will be dampened as $\tilde{\sigma}$ increases. But it is exactly such capital purchases/sales which are the key ingredient to our sunspot volatility, allowing price fluctuations to be self-fulfilled. As $\tilde{\sigma}$ increases, this mechanism is weakened, leading to a decrease in σ_q. Eventually, the mechanism is severed altogether because $\sigma_q^2 < 0$ is not possible.

Steady state stability. In an attempt to differentiate ourselves from the literature, here we examine the traditional stability properties of this model. The addition of idiosyncratic risk provides a convenient environment for stability analysis, for the following reason. Stability properties are typically studied around the "steady state" of a deterministic equilibrium. In Section 2.1 (with $\tilde{\sigma} = 0$), the volatile W-BSE precludes this, and studying a deterministic equilibrium instead puts us in the FE, which trivially has $\kappa = 1$ always. With idiosyncratic risk, we can study a fundamental equilibrium in which capital prices evolve deterministically, even though $\kappa < 1$ in steady state.

The crucial feature of the W-BSE, preserved in this model, is that capital prices are determined by a function q such that $q_t = q(\eta_t)$. Supposing that to be true, a steady state is fully characterized by the value $\eta = \eta^{ss}$ such that all non-growing variables are constant over time. This steady state is thus determined by the equation $\dot{\eta} = 0$, where

$$\dot{\eta} = \eta(1-\eta)\Big[\rho_h - \rho_e + \tilde{\sigma}^2\Big((\frac{\kappa}{\eta})^2 - (\frac{1-\kappa}{1-\eta})^2\Big)\Big] + \delta(\nu - \eta).$$

It is straightforward to show that equilibrium features stable state variable dynamics, in the sense that $\frac{\partial \dot{\eta}}{\partial \eta}|_{\eta=\eta^{ss}} < 0$. However, because the "co-state" q is determined explicitly as a function of η, the steady state is not "stable" in the usual sense required by the multiplicity literature. Technically, there is only one stable eigenvalue of the dynamical system (η_t, q_t) near steady state (η^{ss}, q^{ss}).

Lemma D.3. *The steady state of the model with idiosyncratic risk is saddle path stable.*

Proof of Lemma D.3. First, we show that q is a function of η, i.e., $q_t = q(\eta_t)$. Goods market clearing is still characterized by the price-output relation (PO). With idiosyncratic risk, the risk balance condition (RB) is now (D.2). The solution to the system (PO) and (D.2) can be computed explicitly. Indeed, define

$$\eta^* := \sup\{\eta : (a_e - a_h)\eta\bar{\rho}(\eta) = a_e\tilde{\sigma}^2\}.$$

Then, $\kappa = 1$ for all $\eta \in (\eta^*, 1)$. For $\eta \in (0, \eta^*)$, we compute $\kappa < 1$ as the positive root $\tilde{\kappa}$ from

$$0 = (a_e - a_h)\tilde{\kappa}^2 + [a_h - \eta(a_e - a_h)]\tilde{\kappa} - \eta a_h - \frac{\eta(1-\eta)(a_e - a_h)\bar{\rho}(\eta)}{\tilde{\sigma}^2}.$$

After determining κ for all values of η, capital price q can be computed from (PO), as an explicit function of η.

Given $q_t = q(\eta_t)$, the dynamics of q_t are given by $\dot{q}_t = q'(\eta_t)\dot{\eta}_t$, which only depends on η and not q (notice that $\dot{\eta}_t$ also only depends on η and not q). Consequently, the linearized system near steady state takes the form

$$\begin{bmatrix} \dot{\eta} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} m_1 & 0 \\ m_2 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ q \end{bmatrix}$$

for $m_1, m_2 \neq 0$. The eigenvalues of this system are $m_1 < 0$ and 0. □

As a result of Lemma D.3, there is a unique transition path $(\eta_t, q_t)_{t\geq 0}$ to steady state, given an initial condition η_0. In other words, q_0 is pinned down uniquely. Our sunspot equilibria are not constructed by randomizing over a multiplicity of transition paths that arise due to steady state stability, which is the usual approach (Azariadis, 1981; Cass and Shell, 1983). This can be seen in a relatively transparent way by examining Lemma D.2, which shows how sunspot equilibria can exist in this model (if $\tilde{\sigma}$ is small enough), despite the instability of the steady state.

D.3 Limited commitment as equilibrium refinement

Here, we add a small limited commitment friction, in the spirit of Gertler and Kiyotaki (2010). The result: only equilibria with the property $\lim_{\eta\to 0}\kappa = 0$ survive, similarly to equilibria with a small amount of idiosyncratic risk (Appendix D.2).

Suppose capital holders can abscond with a fraction $\lambda^{-1} \in (0, 1)$ of their assets and renege on repayment of their short-term bonds. After doing this diversion, the capital holder would have net worth $\tilde{n}_{j,t} := \lambda^{-1} q_t k_{j,t}$.

To prevent diversion, bondholders will impose some limitation on borrowing. To see this, note that diversion delivers utility $\log(\tilde{n}_{j,t}) + \xi_t$, where ξ_t is an aggregate process (independent of the identity j of the diverter). For diversion to be sub-optimal, it must be the case that $\log(\tilde{n}_{j,t}) + \xi_t \leq \log(n_{j,t}) + \xi_t$. As a result, bondholders impose the following leverage constraint to ensure non-diversion is incentive compatible:

$$\frac{q_t k_{j,t}}{n_{j,t}} \leq \lambda. \tag{D.3}$$

We will study the equilibrium with constraint (D.3) additionally imposed, and then we will take $\lambda \to \infty$ so that the limited commitment friction is vanishingly small.

Risk balance condition (RB) is now replaced by

$$0 = \min\Big[1-\kappa,\ \lambda\eta - q\kappa,\ \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}|\sigma_R|^2\Big]. \tag{D.4}$$

The most important feature of equation (D.4) is that leverage constrained experts ($\lambda\eta = q\kappa$) must hold less than the full capital stock ($\kappa < 1$).

Condition (D.4) implies that there exists a threshold $\eta^\lambda := \inf\{\eta : \lambda\eta > q\kappa\}$ below which experts' leverage constraints bind. By combining $\lambda\eta = q\kappa$ with condition (PO) for κ, we obtain an explicit formula for the capital price in this region:

$$q = \frac{1}{2}\Big[\frac{a_h}{\bar{\rho}} + \sqrt{(a_h/\bar{\rho})^2 + 4\lambda\eta(a_e - a_h)/\bar{\rho}}\Big], \quad \text{if} \quad \eta \leq \eta^\lambda. \tag{D.5}$$

Taking the limit $\eta \to 0$ in equation (D.5) shows that $q \to a_h/\rho_h$ and thus $\kappa \to 0$. This proves that there is no flexibility for coordination on a worst-case capital price, unlike the leverage-unconstrained economy. The equilibrium is unique along this dimension, coinciding with $\kappa_0 = 0$.

As the limited commitment problem vanishes ($\lambda \to \infty$), the leverage constraint becomes non-binding at all times (formally $\eta^\lambda \to 0$).[35] But along the sequence, $\kappa_0 = 0$ is uniformly required. (And if we focus on equilibria which are Markov in η, the entire equilibrium converges to the W-BSE of Proposition 1.) We collect these results.

Lemma D.4. *Among all equilibria, only those with the property* $\lim_{\eta \to 0} \kappa = 0$ *survive a vanishingly-small limited commitment friction.*

Intuitively, the leverage constraint gives experts an additional motive to sell capital, which forces coordination on maximal selling in response to negative sunspot shocks. Said differently: due to the prospect of violating the leverage constraint, losses incurred

[35] This intuitive property can be shown easily by taking $\lambda \to \infty$ in (D.5). For any fixed $\eta \in (0,1)$, taking this limit implies $q \to \infty$, which is ruled out by price-output relation (PO).

from retaining capital when others are selling is larger than losses incurred from selling capital when others are retaining it. This property is reminiscent of "risk dominant" equilibria being selected by strategic uncertainty (Harsanyi and Selten, 1988; Frankel et al., 2003), but the exact modeling is different here.

D.4 General CRRA preferences

We modify the model of Section 1 by generalizing preferences to the CRRA type. In particular, we replace the $\log(c)$ term in utility specification (3) with the flow consumption utility $c^{1-\gamma}/(1-\gamma)$. For simplicity, we consider no OLG structure ($\delta = 0$), but we continue to allow experts' discount rate to exceed households' ($\rho_e \geq \rho_h$). We impose $\sigma = 0$ so that any non-deterministic equilibrium is a sunspot equilibrium. Finally, we restrict attention to W-BSEs, i.e., those equilibria in which experts' wealth share η is the only state variable.

Equilibrium. The key equation (14) still holds, repeated here for convenience:

$$\Big[1 - (\kappa - \eta)\frac{q'}{q}\Big]\sigma_\eta = 0. \tag{D.6}$$

The sunspot equilibrium is associated with the term in brackets being equal to zero. Unlike with logarithmic preferences, this condition does not pin down $q(\eta)$ function, because we can no longer write $\kappa(q, \eta)$ from the goods market clearing condition: the consumption to wealth ratio is not constant anymore, and depends on agents' value functions.

The value function can be written as $V_i = v_i(\eta)K^{1-\gamma}/(1-\gamma)$ where $v_i(\eta)$ is determined in equilibrium. Then, consumption is $c_i/n_i = (\eta_i q)^{1/\gamma - 1}/v_i^{1/\gamma}$ where η_i corresponds to the wealth share of sector i. Then, goods market clearing becomes

$$q^{1/\gamma}\Big[\Big(\frac{\eta}{v_e}\Big)^{1/\gamma} + \Big(\frac{1-\eta}{v_h}\Big)^{1/\gamma}\Big] = (a_e - a_h)\kappa + a_h. \tag{D.7}$$

Optimal portfolio decisions imply that

$$0 = \min\Big[1-\kappa, \frac{a_e - a_h}{q} - \Big(\frac{v_h'}{v_h} - \frac{v_e'}{v_e} + \frac{1}{\eta(1-\eta)}\Big)(\kappa - \eta)\sigma_q^2\Big]. \tag{D.8}$$

The HJB equation for $i \in \{e, h\}$ has the familiar form $\rho_i V_i = u(c) + \mathbb{E}[\frac{dV_i}{dt}]$, which becomes

$$\rho_i = \frac{(\eta_i q)^{1/\gamma - 1}}{v_i^{1/\gamma}} + \underbrace{\frac{v_i'}{v_i}\mu_\eta + \frac{1}{2}\frac{v_i''}{v_i}\sigma_\eta^2}_{:=\mu^{v,i}} + (1-\gamma)g. \tag{D.9}$$

The dynamics of η satisfy

$$\sigma_\eta = (\kappa - \eta)\sigma_q \tag{D.10}$$

$$\mu_\eta = \eta(1-\eta)\Big(\varsigma_e \frac{\kappa}{\eta}\sigma_q - \varsigma_h \frac{1-\kappa}{1-\eta}\sigma_q + \frac{c_h}{n_h} - \frac{c_e}{n_e}\Big) - \sigma_\eta \sigma_q \tag{D.11}$$

and agent-specific risk prices satisfy

$$\varsigma_e = -\frac{v_e'}{v_e}\sigma_\eta + \frac{\sigma_\eta}{\eta} + \sigma_q \tag{D.12}$$

$$\varsigma_h = -\frac{v_h'}{v_h}\sigma_\eta - \frac{\sigma_\eta}{1-\eta} + \sigma_q. \tag{D.13}$$

A Markov equilibrium is a set of functions: prices $\{q, \sigma_q, \varsigma_e, \varsigma_h\}$, allocation $\{\kappa\}$, value functions $\{v_h, v_e\}$ and aggregate state dynamics $\{\sigma_\eta, \mu_\eta\}$ that solve the system (D.6)-(D.13).

The fundamental equilibrium corresponds to the solution for (D.6) where $\sigma_\eta = 0$, which implies deterministic economic dynamics. Then, the capital price has no volatility ($\sigma_q = 0$), risk prices are zero ($\varsigma_e = \varsigma_h = 0$), and experts hold the entire capital stock ($\kappa = 1$). The capital price is then solved from (D.7), and the value functions satisfy

$$\rho_i = \frac{(\eta_i q)^{1/\gamma - 1}}{v_i^{1/\gamma}} + \frac{v_i'}{v_i}\underbrace{\eta(1-\eta)\left(\frac{c_h}{n_h} - \frac{c_e}{n_e}\right)}_{=\mu_\eta} + (1-\gamma)g.$$

Conversely, the sunspot equilibrium corresponds to the solution for (D.6) with $\frac{q'}{q} = (\kappa - \eta)^{-1}$ (and potentially $\sigma_\eta \neq 0$).

Disaster belief. With logarithmic preferences, we proved that any sunspot equilibrium must satisfy $\sigma_q(0+) = 0$. This allowed us, in Section D.1, to construct sunspot equilibria with $\kappa(0+) = \kappa_0$ for any $\kappa_0 \in [0, 1)$. With CRRA preferences, we attempt to construct the same class of equilibria, with $\sigma_q(0+) = 0$ and $\kappa_0 \in (0, 1)$.

In order to have a non-degenerate stationary distribution, we have the following requirements. Since $\sigma_\eta(0+) = \kappa_0 \sigma_q(0+) = 0$, the state variable avoids the boundary $\{0\}$

if $\mu_\eta(0+) > 0$. Using (D.8) for $\kappa < 1$, we have[36]

$$\frac{a_e - a_h}{q(0+)} = (\varsigma_e(0+) - \varsigma_h(0+))\sigma_q(0+)$$

which allows us to show that[37]

$$\mu_\eta(0+) = \kappa_0 \frac{a_e - a_h}{q(0+)} > 0.$$

In addition, we need $\mu_\eta(\eta^*+) < 0$ where $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$. This requirement should be satisfied for $\rho_e - \rho_h$ sufficiently large.[38]

Numerical solution. We do not provide an existence proof—which involves the existence of a solution to the ODE system—but construct numerical examples. For tractability, the numerical examples are constructed for $\kappa_0 > 0$, which keeps $q'(0+) = q(0+)/\kappa_0$ bounded.[39]

The numerical strategy is the following. Construct a grid $\{\eta_1, \dots, \eta_N\}$ with limit points arbitrarily close to but bounded away from zero and one. Conjecture value functions $v_h(\eta)$ and $v_e(\eta)$. Impose $\kappa(\eta_1) = \kappa_0$ and use (D.7) to solve for $q(\eta_1)$. At each interior grid point, use $q' = q/(\kappa - \eta)$ and (D.7) to solve for $\kappa(\eta)$ and $q(\eta)$ until $\kappa(\eta^*) = 1$. In this region, recover σ_q from (D.8). For $\eta \in (\eta^*, 1]$ impose $\kappa(\eta) = 1$ and $\sigma_q = 0$, and solve capital price from (D.7). The rest of equilibrium objects are calculated directly from the system above. The guesses of the value functions are updated by augmenting the HJBs (D.9) with a time derivative and moving a small time-step backward, as in Brunnermeier and Sannikov (2016). The procedure terminates when the value functions converge to time-independent functions.

In Figure D.2, we plot the equilibrium objects as functions of η, for different levels of risk aversion γ. In Figure D.3, we make the same plots, for different levels of the disaster belief κ_0. Higher risk aversion (higher γ) or more pessimism about disasters (lower κ_0) generates sunspot equilibria featuring lower capital prices and higher volatility.

[36] Note that this implies $\varsigma_e(0+) - \varsigma_h(0+)$ diverges.

[37] This expression also assumes that $\varsigma_h(0+)$ remains bounded. This is a mild assumption since households own all capital.

[38] There is an important distinction between the restriction not to reach $\eta = 0$ and $\mu_\eta(\eta^*+) < 0$. Without the first one, the equilibrium for any $\kappa_0 > 0$ unravels, while without the second one, the equilibrium is still valid, but it has a degenerate stationary distribution at some value $\eta^{ss} > \eta^*$.

[39] With logarithmic utility, we obtain a limiting result in Proposition D.1, that as $\kappa_0 \to 0$, the equilibrium converges to the W-BSE with $\kappa(0) = 0$. With CRRA, we do not prove such a result analytically, but we do observe numerically what looks like convergence as κ_0 becomes small.

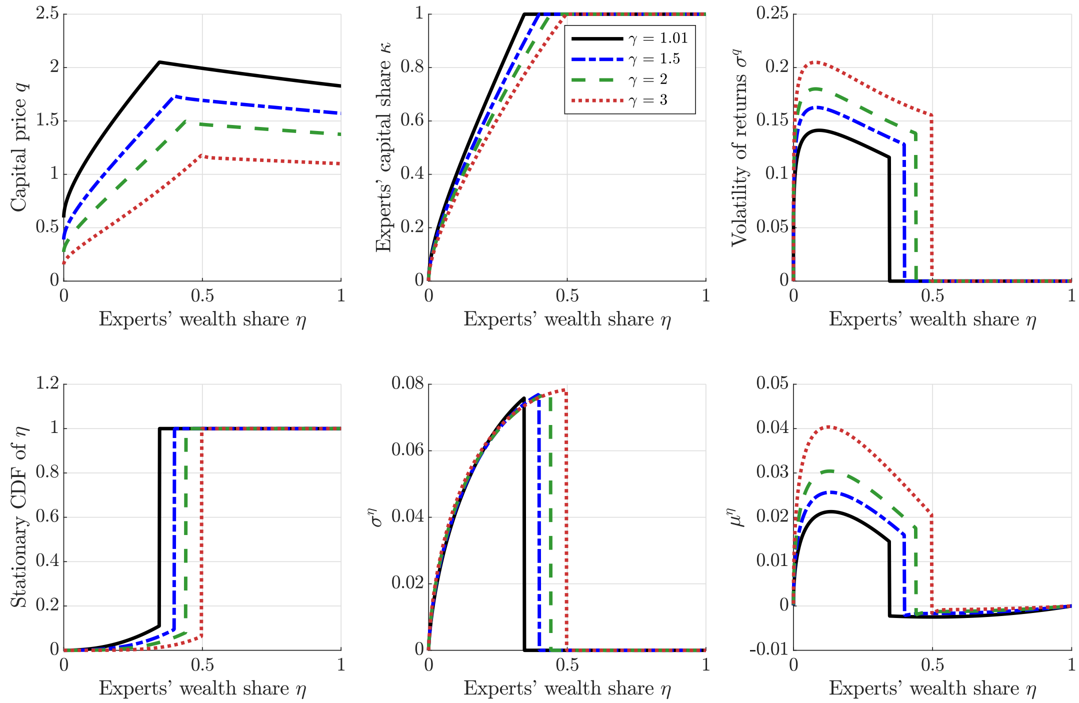

Figure D.2: Sunspot equilibrium for different risk aversion γ. The disaster belief is set to $\kappa_0 = 0.001$. Other parameters: $a_e = 0.11$, $a_h = 0.03$, $\rho_e = 0.06$, $\rho_h = 0.05$, $g = 0.02$.

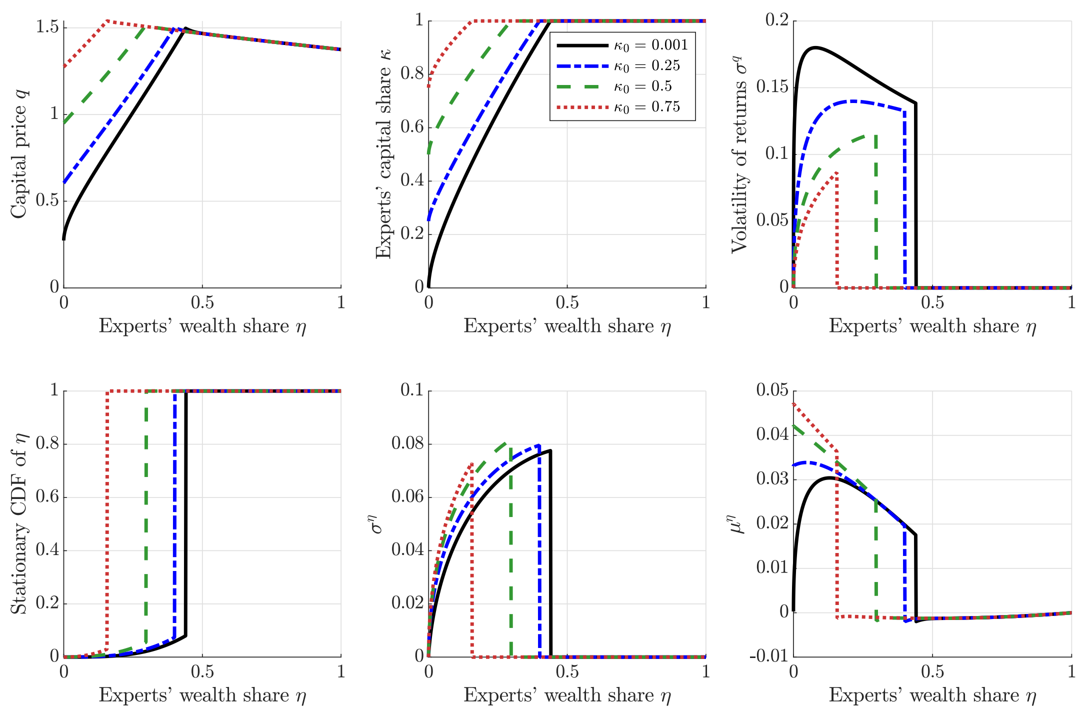

Figure D.3: Sunspot equilibrium for different disaster beliefs κ_0. Risk aversion is set to $\gamma = 2$. Other parameters: $a_e = 0.11$, $a_h = 0.03$, $\rho_e = 0.06$, $\rho_h = 0.05$, $g = 0.02$.

D.5 Correlation between sentiment and fundamentals

What happens if sentiment shocks are correlated with fundamental shocks? To model this, we allow

$$ds_t = \mu_{s,t}dt + \sigma_{s,t}^{(1)}dZ_t^{(1)} + \sigma_{s,t}^{(2)}dZ_t^{(2)}.$$

In Section 4.1, we restricted attention to $\sigma_{s,t}^{(1)} = 0$. Without this assumption, equations (22) and (21) are modified to read:

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa-\eta}{\eta(1-\eta)}\Big(\frac{(\sigma+\sigma_s^{(1)}\partial_s \log q)^2 + (\sigma_s^{(2)}\partial_s \log q)^2}{(1-(\kappa-\eta)\partial_\eta \log q)^2}\Big)\right]$$
$$\sigma_q = \frac{\binom{1}{0}(\kappa-\eta)\sigma\partial_\eta \log q + \sigma_s \partial_s \log q}{1-(\kappa-\eta)\partial_\eta \log q}.$$

The rest of the equilibrium restrictions are identical.

For the present illustration, we additionally assume that $\sigma_{s,t}^{(2)} = 0$, i.e., sentiment shocks *only* load on fundamental shocks. What emerges is the possibility that sentiment shocks "hedge" fundamental shocks: we can have $\sigma_s^{(1)}\partial_s \log q < 0$, which lowers return volatility and raises asset prices. In the extreme case, if $\sigma_s^{(1)}\partial_s \log q \to -\sigma$, the economy will converge to the W-BSE of Section 2.1. At the other end, if $\sigma_s^{(1)}\partial_s \log q \to 0$, the economy resembles the Fundamental Equilibrium (FE) with positive fundamental shocks (this FE was q^{FE} in our baseline construction in Section 4.1). Thus, by constructing our conjectured capital price function as a convex combination of the W-BSE and the FE, with weight $1-s$ on the W-BSE and s on the FE, we can ensure that $\sigma_s^{(1)}\partial_s \log q$ endogenously emerges negative. Figure D.4 displays the equilibrium constructed this way.

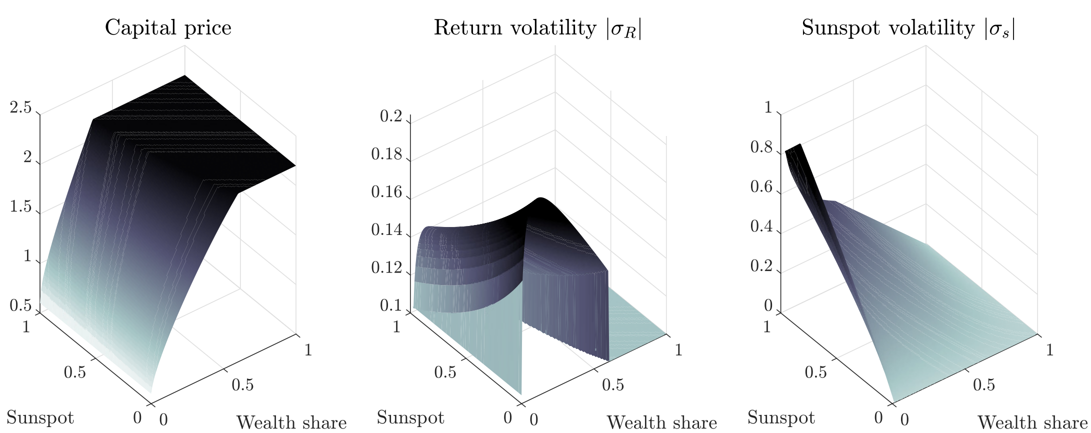

Figure D.4: Capital price q, volatility of capital returns $|\sigma_R|$, and sunspot shock volatility $|\sigma_s|$. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.10$.

D.6 Exogenous sunspot dynamics

In Section 4.1, we solved for a Markov S-BSE that featured endogenous sunspot dynamics, i.e., (σ_s, μ_s) could potentially depend on η. Here, we show that sunspot equilibria can be built on top of exogenous sunspot dynamics as well. As we will show, this construction can be naturally viewed as the limit of equilibria in which the variable s has a vanishing contribution to fundamentals. With that in mind, we actually start from a more general setting in which s can impact fundamental volatility, and then we take the limit as this impact becomes vanishingly small.

Consider the following stochastic volatility model:

$$\begin{aligned}\frac{dK_t}{K_t} &= gdt + \sigma\sqrt{1+\omega s_t}dZ_t \\ ds_t &= \mu_s(s_t)dt + \vartheta\sqrt{1+\omega s_t}dZ_t\end{aligned}$$

where $\vartheta > 0$ is an exogenous parameter and $\omega \in \mathbb{R}$ measures the impact of s_t on capital growth volatility. Thus, the diffusion of s_t, namely $\sigma_s(s) := \vartheta\sqrt{1+\omega s}$, is specified exogenously. Also, $\mu_s(s)$ is an exogenous function that is specified to ensure that $s_t \in (s_{\min}, s_{\max})$, for some pre-specified interval satisfying $s_{\min} \geq 0$ and $cs_{\max} > -1$. Such a choice can always be made, e.g., by putting $\mu_s(s) = -(s_{\max} - s)^{-(1+\beta)} + (s - s_{\min})^{-(1+\beta)}$. Note that s_t becomes a sunspot when $\omega = 0$. When $\omega < 0$, the state s_t is an inverse measure of capital's volatility.

For simplicity, we assume there is a single aggregate shock, i.e., Z is a one-dimensional Brownian motion; this can easily be generalized to multiple shocks. Also for simplicity of expressions, we assume here that $\rho_e = \rho_h = \rho$. Then, an equilibrium capital price function $q(\eta, s)$ must satisfy the PDE defined by the following system

$$\begin{aligned}\rho q &= \kappa a_e + (1-\kappa)a_h \\ 0 &= \min\left[1-\kappa,\, \frac{a_e - a_h}{q} - \frac{(\kappa-\eta)(1+\omega s)}{\eta(1-\eta)}\Big(\frac{\sigma + \vartheta\partial_s \log q}{1-(\kappa-\eta)\partial_\eta \log q}\Big)^2\right].\end{aligned}$$

Technically, the multiplicity arises from the selection of the boundary conditions on $q(\eta, s_{\min})$ and $q(\eta, s_{\max})$, which are not pinned down by any equilibrium restriction.

We perform two exercises. First, we show that there are multiple equilibria for a given set of parameters. We use $\omega < 0$ here, along with $s_{\min} = 0$ and $s_{\max} = 2$. In this case, the "natural" and intuitive solution is for q to increase with s, while volatility decreases. In Figure D.5, we pick a "low" boundary condition for $q(\eta, 0)$ and the solution follows this intuition.[40]

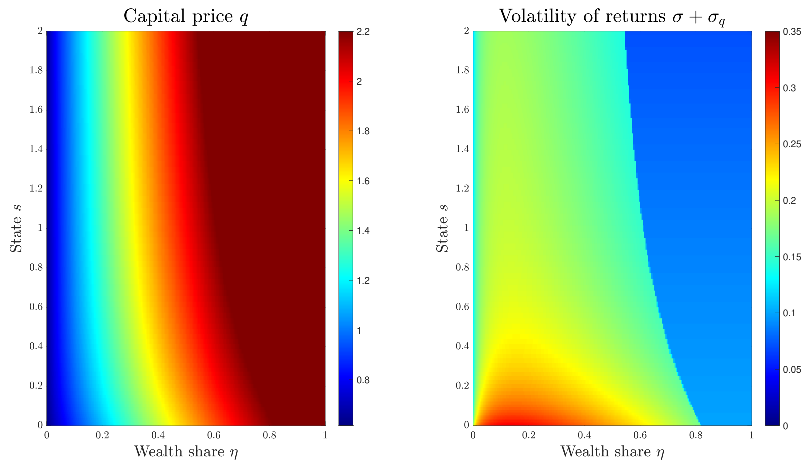

Figure D.5: Equilibrium with $\omega = -0.25$, and the "low" boundary condition for $q(\eta, 0)$, which is a 50% weighted-average of the fundamental equilibrium and the infinite-volatility equilibrium. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$, $\vartheta = 0.25$. The boundary condition at $\eta = 0$ is set so that $\kappa(0, s) = 0.01$ for all s.

However, agents could equally well coordinate on a "high" boundary condition, which results in the solution of Figure D.6.[41] Notice the capital price and return volatility exhibit a non-monotonicity in s. At low values of s, q is decreasing in s, while return volatility increases. This behavior is made possible by the "coordination component" of the response to changes in s and not by the "fundamental component."

Our second exercise considers the limit $\omega \to 0$. Figure D.7 shows the solution for $\omega = -10^{-6}$, again equipped with the "low" boundary condition for $q(\eta, 0)$. There remains a tremendous amount of variation in the equilibrium as s varies, illustrating convergence to a sunspot equilibrium. Thus, as promised, we are able to construct sunspot equilibria even if the dynamics (σ_s, μ_s) are specified exogenously. In fact, it appears that the amount of price volatility is relatively insensitive to the real effects s has (i.e., the size of ω), which is reminiscent of the "volatility paradox" of Brunnermeier and Sannikov (2014) but one level deeper. Their paradox is that total volatility is only modestly sensitive to exogenous fundamental volatility; our paradox is that total volatility is only modestly sensitive to the *exogenous impact of s on fundamental volatility*.

[40]This "low" boundary condition is a weighted average between the solution with infinite volatility and the fundamental equilibrium solution. The fundamental equilibrium, which is the capital price solution that keeps $s = 0$ fixed forever, is discussed in Online Appendix E. The infinite-volatility solution has $\kappa = \eta$, hence $q = (\eta a_e + (1 - \eta) a_h) / \bar{\rho}(\eta)$.

[41]This "high" boundary condition is a weighted average between the W-BSE of Section 2.1 (which is a potential solution to the equilibrium with $\sigma = 0$) and the fundamental equilibrium solution.

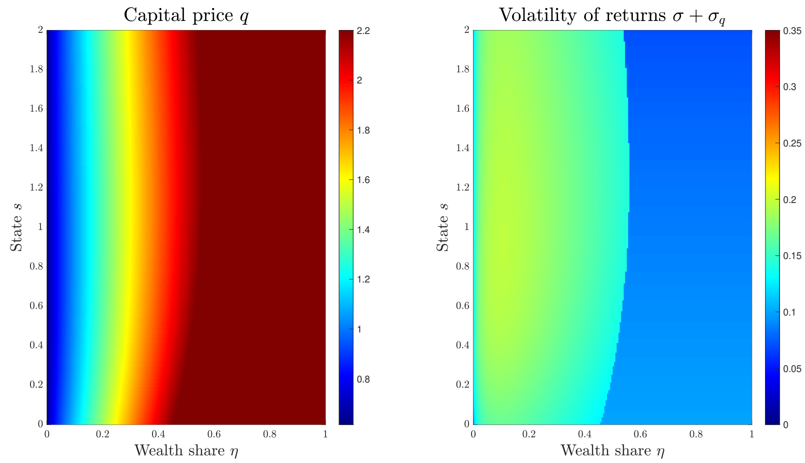

Figure D.6: Equilibrium with $\omega = -0.25$, and the "high" boundary condition for $q(\eta, 0)$, which is a 50% weighted-average of the fundamental equilibrium and a W-BSE. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$, $\vartheta = 0.25$. The boundary condition at $\eta = 0$ is set so that $\kappa(0, s) = 0.01$ for all s.

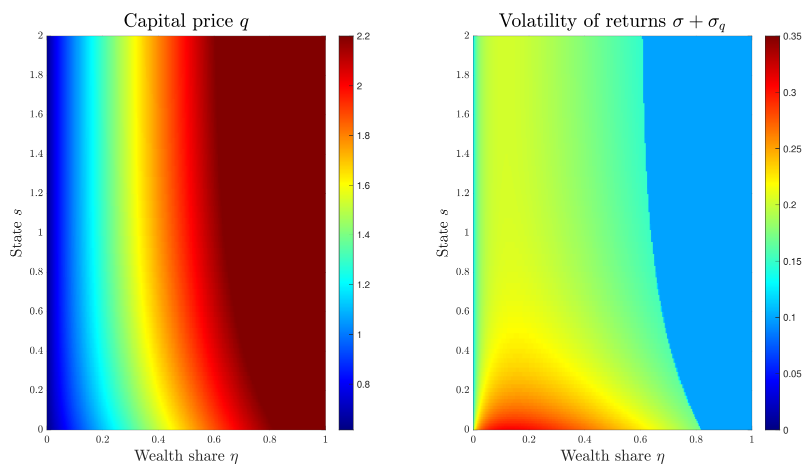

Figure D.7: Equilibrium with near-sunspot $\omega = -10^{-6}$ and the "low" boundary condition for $q(\eta, 0)$, which is a 50% weighted-average of the fundamental equilibrium and the infinite-volatility equilibrium. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$, $\vartheta = 0.25$. The boundary condition at $\eta = 0$ is set so that $\kappa(0, s) = 0.01$ for all s.

E Fundamental Equilibria

In this section, we investigate properties of equilibria where sunspot shocks $Z^{(2)}$ are irrelevant and experts' wealth share η serves as the only state variable, i.e., fundamental equilibria. We illustrate previously undocumented multiplicity along two dimensions: the disaster belief κ_0 and the sign of the sensitivity of capital returns to fundamental shocks $\sigma + \sigma_q$. The key equations describing FEs are (PO), (A.1), and (18), restated here for convenience:

$$q\bar{\rho} = \kappa a_e + (1-\kappa)a_h \tag{E.1}$$

$$0 = \min\left[1-\kappa,\ \frac{a_e - a_h}{q} - \frac{\kappa-\eta}{\eta(1-\eta)}(\sigma+\sigma_q)^2\right]. \tag{E.2}$$

$$\sigma_q = \frac{(\kappa-\eta)q'/q}{1-(\kappa-\eta)q'/q}\sigma. \tag{E.3}$$

Also, wealth share dynamics are given in (11)-(12), restated here for convenience:

$$\mu_\eta = -\eta(1-\eta)(\rho_e-\rho_h) + \mathbf{1}_{\{\kappa<1\}}(\kappa - 2\kappa\eta+\eta^2)\frac{a_e-a_h}{q} + \delta(\nu-\eta) \tag{E.4}$$

$$\sigma_\eta = (\kappa-\eta)(\sigma+\sigma_q). \tag{E.5}$$

We define a fundamental equilibrium as follows.[42]

Definition 5. Given $\eta_0 \in (0,1)$, a *Markov fundamental equilibrium* consists of adapted processes $(\eta_t, q_t, \kappa_t)_{t\geq 0}$ such that (E.1)-(E.3) hold, and (E.4)-(E.5) describe dynamics of η_t.

E.1 Properties of the non-sunspot solution with fundamental shocks

We describe here some properties of fundamental equilibria with fundamental volatility $\sigma > 0$, where we additionally impose the full-deleveraging condition $\kappa(0) = 0$.

Lemma E.1. *Assuming it exists, suppose (q,κ) is a fundamental equilibrium in η in the sense of Definition 5. Assume $\kappa(0+) = 0$. Define $\eta^* := \inf\{\eta : \kappa = 1\}$. Then, the following hold:*

(i) $(\bar{\rho}q - \eta a_e - (1-\eta)a_h)\frac{q'}{q} = a_e - a_h - \sigma\sqrt{q\frac{\bar{\rho}q-\eta a_e-(1-\eta)a_h}{\eta(1-\eta)}}$*, for all* $\eta \in (0,\eta^*)$.

(ii) $\eta a_e + (1-\eta)a_h < \bar{\rho}q < a_e$*, for all* $\eta \in (0,\eta^*)$.

(iii) $\frac{q'(0+)}{q(0+)} = \frac{a_e}{a_h} - \frac{\rho_e}{\rho_h} + \rho_h\left(\frac{a_e-a_h}{\sigma a_h}\right)^2$.

(iv) If σ is sufficiently small, then $q' > \frac{a_e-a_h}{\bar{\rho}}$*, for* $\eta \in (0,\eta^*)$.

(v) If σ is sufficiently small, then $\frac{\rho_h}{\rho_e}\left(\frac{1-a_h/a_e}{\sigma^2} - 1 + \frac{\rho_h}{\rho_e}\right)^{-1} < \eta^* < 1$.

(vi) On $\eta \in (0,\eta^*)$, the solution q is infinitely-differentiable.

[42]We omit r_t from the definition, since it can be read off of (9), given other objects, and affects no other equation.

Proof of Lemma E.1. Since a fundamental equilibrium is assumed to exist, we make use of equations (E.1) and (E.2). Recall that $\bar{\rho} := \eta\rho_e + (1-\eta)\rho_h$. By analogy, let $\bar{a} := \eta a_e + (1-\eta)a_h$.

(i) Start from equation (E.2), and rearrange to obtain the result, where we have implicitly selected the solution with $1 > (\kappa - \eta)\frac{q'}{q}$.

(ii) The first inequality, which is equivalent to $\kappa > \eta$, is a direct implication of equation (E.2). The second inequality, equivalent to $\kappa < 1$, is a restatement of the definition of η^*.

(iii) Start from equation (E.2). Taking the limit $\eta \to 0$, and using $\kappa(0+) = 0$, delivers an equation for $\kappa'(0+)$. Differentiating (E.1), we may then substitute $\kappa'(0+) = \frac{\rho_h q'(0+) + (\rho_e - \rho_h) q(0+)}{a_e - a_h}$. Rearranging, we obtain the desired result.

(iv) By part (iii), there exists $\eta^\circ > 0$ and $\bar{\sigma} > 0$ such that uniformly for all $\sigma < \bar{\sigma}$, we have $q' > \frac{a_e - a_h}{\bar{\rho}}$ on the set $\{\eta < \eta^\circ\}$. On the set $\{\eta^\circ \leq \eta < \eta^*\}$, we know that $\kappa - \eta$ is bounded away from zero, uniformly for all $\sigma < \bar{\sigma}$. Using the expression in part (i), the fact that q is bounded by $a_e/\bar{\rho}$ uniformly for all σ, and the previous fact about $\kappa - \eta = \bar{\rho}q - \bar{a}$, we can write

$$q' = \frac{a_e - a_h}{\bar{\rho}q - \bar{a}} q - o(\sigma), \quad \eta \in (\eta^\circ, \eta^*).$$

Therefore,

$$q' + o(\sigma) = \frac{a_e - a_h}{\bar{\rho}q - \bar{a}} q = \frac{a_e - a_h}{\bar{\rho}} \frac{q}{q - \bar{a}/\bar{\rho}} > \frac{a_e - a_h}{\bar{\rho}}, \quad \eta \in (\eta^\circ, \eta^*),$$

where the last inequality is due to $\bar{\rho}q > \bar{a}$ [part (ii)]. Taking σ is small enough implies the result on (η°, η^*), which we combine with the result on $(0, \eta^\circ)$ to conclude.

(v) Consider the function $\tilde{q} := \bar{a}/\bar{\rho}$, whose derivative is $\tilde{q}' = \frac{a_e - a_h}{\bar{\rho}} - \frac{\bar{a}}{\bar{\rho}} \frac{\rho_e - \rho_h}{\bar{\rho}} < \frac{a_e - a_h}{\bar{\rho}}$. Combining this result with part (iv), we obtain $q' > \tilde{q}'$. If $\tilde{q}$ was the capital price, then equation (E.1) implies the associated capital share $\tilde{\kappa} = \eta$. On the other hand, the fact that $q' > \tilde{q}'$ implies $\kappa' > \tilde{\kappa}' = 1$, which implies $\eta^* < 1$.

Next, consider $\eta \in (\eta^*, 1)$ so that $\kappa = 1$. By equation (E.2), with $q = a_e/\bar{\rho}$, we must have

$$\sigma^2 \leq \eta\bar{\rho} \frac{a_e - a_h}{a_e} \Big(1 + (1-\eta)\frac{\rho_e - \rho_h}{\bar{\rho}}\Big)^2, \quad \eta \geq \eta^*.$$

This is equivalent to

$$1 \leq \eta \frac{\rho_e}{\rho_h} \Big(\frac{a_e - a_h}{a_e \sigma^2} \rho_e - 1 + \frac{\rho_h}{\rho_e} \Big), \quad \eta \geq \eta^*.$$

Substituting $\eta = \eta^*$, and rearranging, we obtain the first inequality. There is no contradiction with $\eta^* < 1$, due to the assumption that σ can be made small enough.

(vi) Note that $F(\eta, q) := q[\frac{a_e - a_h}{\bar{\rho}(\eta)q - \bar{a}(\eta)} - \sigma(\frac{\eta(1-\eta)(\bar{\rho}(\eta)q - \bar{a}(\eta))}{q})]$ is infinitely differentiable in both arguments on $\{(\eta, q) : \eta \in (0,1), \bar{\rho}(\eta)q > \bar{a}(\eta)\}$. Thus, the result is a simple consequence of differentiating part (i), noting that by part (ii) we have $\bar{\rho}(\eta)q(\eta) > \bar{a}(\eta)$, and then using induction. $\square$

Although the existing literature always imposes $\kappa(0+) = 0$, this is actually not a necessary feature of a fundamental equilibrium.[43] If we let $\kappa_0 \in (0,1)$ be a given "disaster belief" about experts' deleveraging and we suppose $\kappa(0+) = \kappa_0$ (similar to Appendix D.1 for the sunspot case with $\sigma = 0$), there is no inherent contradiction to equilibrium. Existence of such an equilibrium boils down simply to existence of a solution to a first-order ODE. Thus, a variety of fundamental equilibria could exist, and indeed we provide a numerical example after the following lemma and proof.

Lemma E.2. *A fundamental equilibrium with disaster belief $\kappa_0 \in (0,1)$ exists if the free boundary problem*

$$(\bar{\rho}q - \eta a_e - (1-\eta)a_h)\frac{q'}{q} = a_e - a_h - \sigma\sqrt{q\frac{\bar{\rho}q - \eta a_e - (1-\eta)a_h}{\eta(1-\eta)}}, \quad \text{on} \quad \eta \in (0, \eta^*), \tag{E.6}$$

$$\text{subject to} \quad q(0) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h} \quad \text{and} \quad q(\eta^*) = \frac{a_e}{\bar{\rho}(\eta^*)}, \tag{E.7}$$

has a solution.

[43] Brunnermeier and Sannikov (2014) justify $\kappa_0 = 0$ in their online appendix: "because in the event that η_t drops to 0, experts are pushed to the solvency constraint and must liquidate any capital holdings to households." This is technically not needed; as shown in Lemma E.2 of Appendix E.1, the dynamics of η_t will not allow it to ever reach 0, so there is no contradiction to equilibrium with both $\kappa_0 > 0$ and $\sigma > 0$. Although we do not prove an existence result, Appendix E.1 presents several numerical examples. The continuum of fundamental equilibria, indexed by κ_0, may be of independent theoretical interest.

In some sense, the literature has picked the worst possible fundamental equilibrium (minimal-price, maximal-volatility) by imposing $\kappa_0 = 0$. This can be partly justified by the refinement results of Sections D.2 and D.3, which carry over to the case with $\sigma > 0$, i.e., only the belief $\kappa_0 = 0$ survives vanishingly-small idiosyncratic risk or a vanishingly-small limited commitment friction.

Proof of Lemma E.2. A fundamental equilibrium in state variable η exists if and only if equations (E.1), (E.2), and (E.3) hold, and if the time-paths $(\eta_t)_{t\geq 0}$ induced by dynamics (σ_η, μ_η) avoid $\eta = 0$ almost-surely. We will demonstrate these conditions.

Suppose (E.6)-(E.7) has a solution (q, η^*) corresponding to $\kappa_0 \in (0,1)$. If there are multiple solutions, we pick the one such that $q(\eta) < a_e/\bar{\rho}(\eta)$ for all $\eta \in (0, \eta^*)$, which is always possible because the boundary conditions (E.7) imply $\bar{\rho}(0)q(0) < \bar{\rho}(\eta^*)q(\eta^*)$. Set $q(\eta) = a_e/\bar{\rho}(\eta)$ for all $\eta \geq \eta^*$. Define $\kappa = \frac{\bar{\rho}q - a_h}{a_e - a_h}$. Note that (E.1) is automatically satisfied. Note that (E.3) is also satisfied automatically, by applying Itô's formula to the solution $q(\eta)$ and using $\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q)$.

We show (E.2) holds separately on $(0, \eta^*)$ and $[\eta^*, 1)$. Using (E.1) and (E.3) in the ODE (E.6) and rearranging, we show that (E.2) holds when $\kappa < 1$. The boundary condition $q(\eta^*) = a_e/\bar{\rho}(\eta^*)$ is equivalent to $\kappa(\eta^*) = 1$, which shows that $\kappa(\eta) < 1$ for all $\eta < \eta^*$. Therefore, we have proven that (E.2) holds on $(0, \eta^*)$.

If $\eta^* = 1$, then we are done verifying (E.2); otherwise, we need to verify (E.2) on $[\eta^*, 1)$. On this set, $\kappa = 1$, so we need to verify

$$\eta \frac{a_e - a_h}{q} \geq (\sigma + \sigma_q)^2 \quad \text{for all} \quad \eta \geq \eta^*. \tag{E.8}$$

First, we show that it suffices to verify this condition exactly at η^*. Indeed, on $(\eta^*, 1)$, we have $\kappa = 1$ and $q = a_e/\bar{\rho}$. Substituting these and (E.3) into (E.8), we obtain

$$\text{(E.8) holds} \;\Leftrightarrow\; \Big(\frac{a_e - a_h}{a_e \sigma^2}\rho_e - \frac{\rho_e - \rho_h}{\rho_e}\Big)\eta \geq \frac{\rho_h}{\rho_e} \quad \text{for all} \quad \eta \geq \eta^*.$$

But since the left-hand-side is increasing in η, if it holds at $\eta = \eta^*$, it holds for all $\eta > \eta^*$.

Now, writing (E.8) at η^*, using (E.3) to replace σ_q, and using ODE (E.6) to replace $\eta^* \frac{a_e - a_h}{q(\eta^*)} = \sigma[1 - (1-\eta^*)q'(\eta^*-)/q(\eta^*)]^{-1}$, we need to verify

$$\text{(E.8) holds} \;\Leftrightarrow\; \frac{\sigma}{1 - (1-\eta^*)q'(\eta^*-)/q(\eta^*)} \geq \frac{\sigma}{1 - (1-\eta)q'(\eta^*+)/q(\eta^*)} \;\Leftrightarrow\; q'(\eta^*-) \geq q'(\eta^*+).$$

We clearly have $q'(\eta^*-) \geq q'(\eta^*+)$ by the simple fact that $q < a_e/\bar{\rho}$ for $\eta < \eta^*$ and $q = a_e/\bar{\rho}$ for $\eta \geq \eta^*$.

Finally, it remains to very that η_t almost-surely never reaches the boundary 0. Near

$\eta = 0$, the dynamics in (E.4)-(E.5) become

$$\mu_\eta(\eta) = \kappa_0 \frac{a_e - a_h}{q(0+)} + \delta\nu + o(\eta)$$
$$\sigma_\eta^2(\eta) = \kappa_0 \frac{a_e - a_h}{q(0+)} \eta + o(\eta).$$

By the same analysis as in Theorem 1, the boundary 0 is unattainable. □

What happens in an equilibrium of Lemma E.2 in which $\kappa_0 > 0$? Behavior at the boundary $\eta = 0$ is substantially different than the $\kappa_0 = 0$ case, because equation (E.2) can only hold there if $\sigma_q \to -\sigma$ as $\eta \to 0$. Capital prices "hedge" fundamental shocks to capital, in a brief region of the state space $(0, \eta^{\text{hedge}})$. Said differently, given the formula (E.3), the fact that $\sigma_q(0+) = -\sigma$ implies $q'(0+) = -\infty$, so that prices rise as experts lose wealth in a region of the state space. The hedging region is exactly what incentivizes experts to take so much leverage (indeed, expert leverage κ/η blows up near 0). For $\eta > \eta^{\text{hedge}}$, this behavior reverses, and the equilibrium behaves very much like the equilibrium with $\kappa_0 = 0$. Overall, there is no inconsistency with equilibrium even though $q' < 0$ in the region $(0, \eta^{\text{hedge}})$.[44]

Figure E.1 displays several examples of equilibria with different choices of $\kappa_0 > 0$. The solid black lines, which are equilibrium outcomes with $\kappa_0 = 0.001$, corresponds approximately to the equilibrium choice made by Brunnermeier and Sannikov (2014). The other curves, with higher disaster beliefs κ_0, are new to the literature. Similar to the the sunspot results of Section D.1, more optimistic disaster beliefs raise capital prices and reduce capital price volatility.

E.2 The "hedging" equilibrium

The equilibria described in Appendix E.1 are "normal" in the sense that a positive exogenous shock increases asset prices and experts' wealth share. But technically, agents do not care about the direction prices move when they make their portfolio choices. They only care about risk which is measured in return variance; this can be seen in the

[44] One may think that $q'(0+) = -\infty$, and more generally that $q' < 0$ in some region of the state space, could imply that κ hits η at some point. However, this cannot happen. Indeed, since $\kappa_0 > 0$, we have that $q(0+) > \tilde{q}(0+)$, where $\tilde{q}(\eta) := ((a_e - a_h)\eta + a_h)/\bar{\rho}$ is the price function consistent with $\kappa = \eta$.

To see this, assume there is an $\hat{\eta} \in (0,1)$ such that $\kappa(\hat{\eta}) = \hat{\eta}$ (or equivalently, $q(\hat{\eta}) = \tilde{q}(\hat{\eta})$). If there is more than one, consider the minimum among them, so $q(\eta) > \tilde{q}(\eta)$ for all $\eta \in (0, \hat{\eta})$. From the $\tilde{q}(\eta)$ definition, we have $\tilde{q}'(\eta) = (a_e - a_h)/\bar{\rho} - ((a_e - a_h)\hat{\eta} + a_h)(\rho_e - \rho_h)/\bar{\rho}^2 < \infty$, while from (E.6) it must be that $q'(\hat{\eta}-) \to \infty$. This is a contradiction.

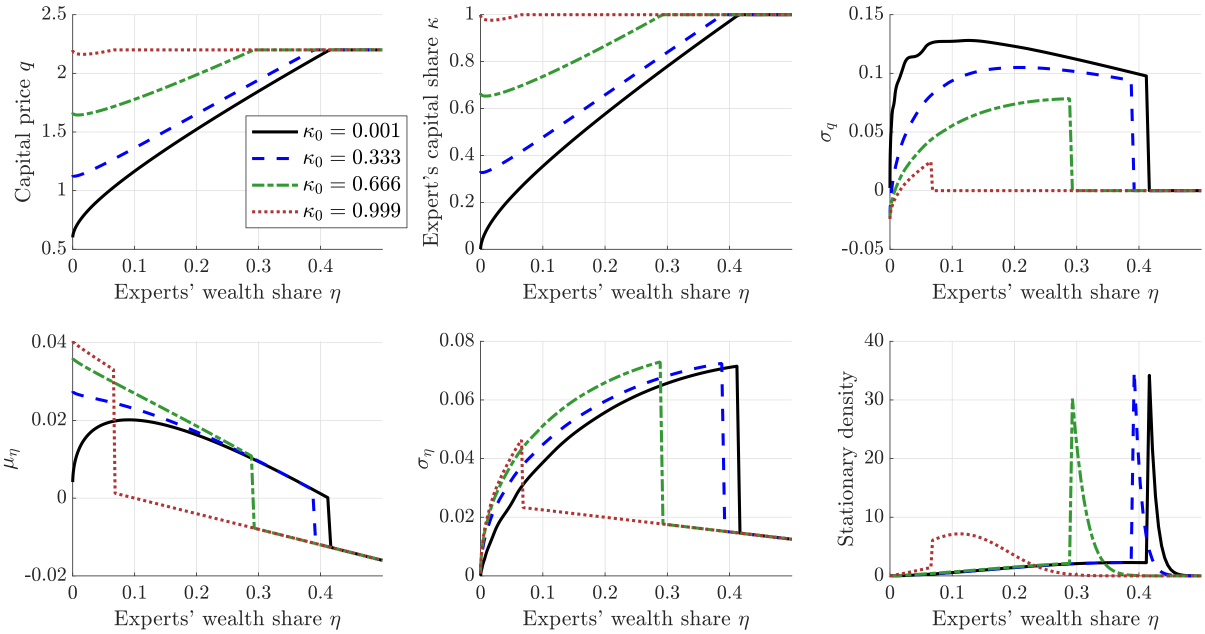

Figure E.1: Fundamental equilibria with different disaster beliefs κ_0. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$.

optimality condition (E.2) where $(\sigma + \sigma_q)^2$ appears. An immediate implication is that two types of equilibria are possible: the "normal" one has $\sigma + \sigma_q > 0$; an alternative equilibrium has $\sigma + \sigma_q < 0$.[45]

We term this latter equilibrium the "hedging" equilibrium because asset price movements move oppositely to exogenous shocks. In fact, asset price responses are so strong in opposition that experts actually gain in wealth share upon a negative fundamental shock. This can only happen because of coordination: experts and households simply believe negative shocks are good news for asset prices, so they rush to purchase capital, which percolates through equilibrium relationships to justify beliefs about price increases. Such coordination stands in contrast to the normal equilibrium, in which negative shocks beget fire sales that push down asset prices.

Mathematically, we need only solve a slightly different capital price ODE. Whereas ODE (E.6) holds in the normal equilibrium, the hedging equilibrium requires

$$(\bar{\rho}q - \eta a_e - (1-\eta)a_h)\frac{q'}{q} = a_e - a_h + \sigma\sqrt{q\frac{\bar{\rho}q - \eta a_e - (1-\eta)a_h}{\eta(1-\eta)}}, \quad \text{on} \quad \eta \in (0, \eta^*). \tag{E.9}$$

The difference between (E.9) and (E.6) is merely the sign in front of $\sigma + \sigma$, which ensures different signs for σ_q. Finally, note that just like the normal equilibria, hedging

[45]For a conjecture of this specific type of indeterminacy, see footnote 16 of Kiyotaki and Moore (1997).

equilibria could exist for $\kappa_0 \neq 0$. Figure E.2 compares a normal equilibrium to a hedging equilibrium.

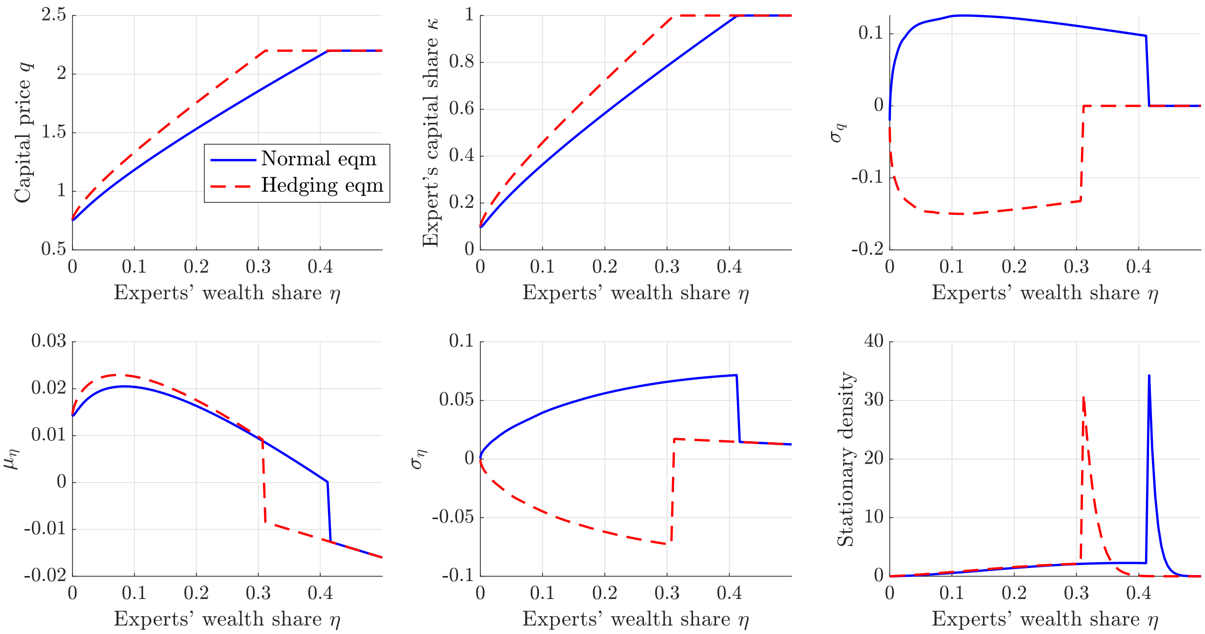

Figure E.2: Two equilibria (normal versus hedging) both with disaster belief $\kappa_0 = 0.1$. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$.